A hybrid *level-set*/*embedded boundary* numerical method for the simulation of solidification/melt problems

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Solidification/melt problems

Figure: Ice falling from bridge cables

Figure: Drone de-icing of a wind turbine

Figure: Power outage due to ice

Figure: Iced wing

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How can we properly do numerical simulation of solidification/melt problems ?

Level set method

- ϕ a higher-dimensional function
- Interface Γ: zero level-set of ϕ (hypersurface)[1].

In our cases, ϕ is initialized as a signed distance function.

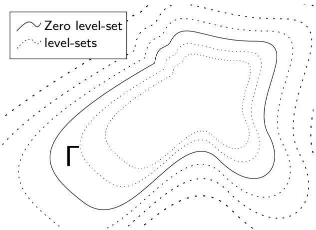

Figure: Interface Γ and different level-sets of ϕ

[1] S. Osher and J. A. Sethian. "Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations". In: *Journal of computational physics* 79.1 (1988), pp. 12–49.

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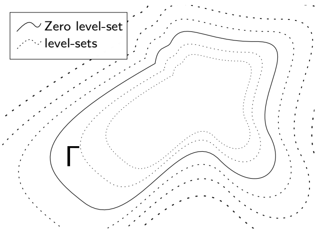
$$\partial_t \phi + \mathbf{v_{pc}} \cdot \nabla \phi = 0. \qquad (1)$$

$\mathbf{v_{pc}}$: phase change velocity field.



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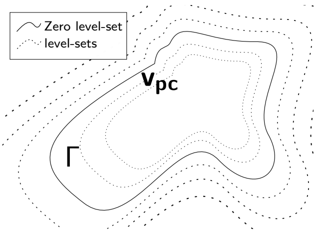

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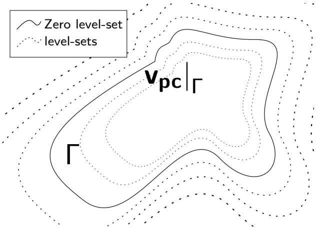
$\mathbf{v_{pc}}$: phase change velocity field.
$\Rightarrow$ Construct a continuous $\mathbf{v_{pc}}$ field using $\mathbf{v_{pc}}\big|_\Gamma$ and ϕ



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Reconstruction of the velocity

- Extrapolation[2] of $\mathbf{v_{pc}}|_\Gamma$ in the vicinity of the interface:

$$\frac{\partial v_{pc}}{\partial t} + \text{sign}(\phi)\left(\mathbf{n} \cdot \nabla v_{pc}\right) = 0$$

with:

$$\mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

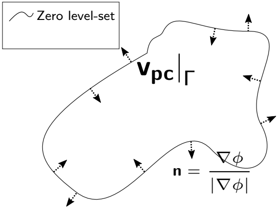

Figure: Local advection of $\mathbf{v_{pc}}$

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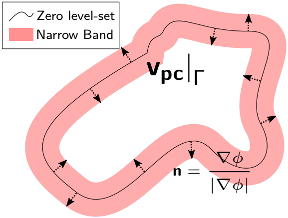
$\Rightarrow$ Narrow band approximation.



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Redistancing of ϕ

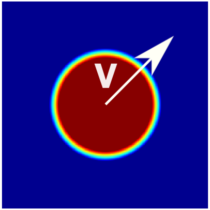
Issue

- Advection $\rightarrow$ numerical diffusion

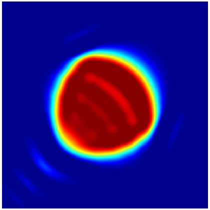
$\Rightarrow$ correction method is required
$\Rightarrow$ reinitialization/redistancing of ϕ^0:

$$\frac{\partial \phi}{\partial t} = \operatorname{sgn}(\phi^0)\,(1 - \nabla\phi)$$

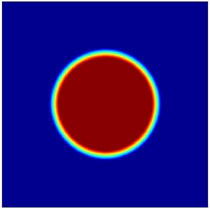
Solved only in the Narrow Band[3]



(a) Initial ϕ field



(b) Final ϕ w/o redistancing



(c) Final ϕ w/ redistancing

Figure: Redistancing required

[3] Giovanni Russo and Peter Smereka. "A remark on computing distance functions". In: *Journal of Computational Physics* 163.1 (2000), pp. 51–67.

Level-set methods

Pros

- Simple to implement
- Made for flows with complex interfacial topologies; splitting/merging, shear...

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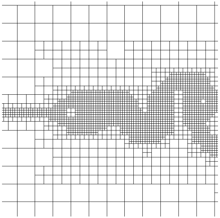
Cons

- Non-conservative (reinitialization step)

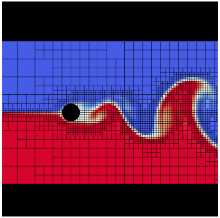
Embedded Boundaries

Concept

- Intersect a boundary of general shape with a grid (here Cartesian)
- Modify volume and area fractions of the intersected cells (Finite Volume formulation)



(a) Grid



(b) Passive tracer with an cylindrical embedded boundary

Figure: Flow past a circle with embedded boundaries

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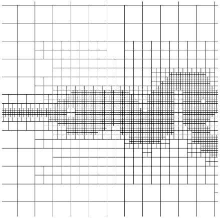

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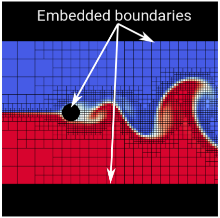

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Embedded Boundaries

Gradients on the interface

Johansen and Colella's method[4]:

$$\nabla\phi|_f = \frac{1}{d_2 - d_1}\left(\frac{d_2}{d_1}(\phi_f - \phi_1) - \frac{d_1}{d_2}(\phi_f - \phi_2)\right)$$

second-order accurate

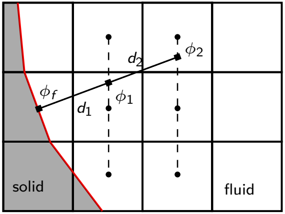

Figure: embedded boundary principle

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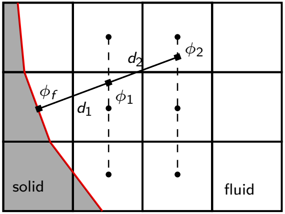
Non-moving boundaries, or solid movement using ALE formulation in its original formulation !



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Main idea

- 2 fluids : Calculation inside and outside of the boundary with two set of fields
- v_{pc} only field defined in both phases with a meaning

We use the gradients defined by the high-order approximation of the embedded boundary method and set:

Stefan condition:
$$\mathbf{v}_{pc}|_\Gamma = \frac{1}{L_H}\left(\frac{\lambda_L}{\rho_L}\nabla T_L|_\Gamma - \frac{\lambda_S}{\rho_S}\nabla T_S|_\Gamma\right)$$

Gibbs-Thomson equation:
$$T_\Gamma = T_m - \epsilon_\kappa * \kappa - \epsilon_v * v_{pc}$$

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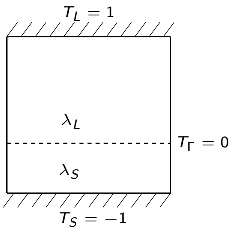
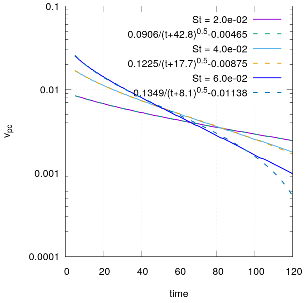
- Possible use of 2 different solvers

1D solidification - diffusion



(a) Schematic of the case



(b) $\mathbf{v_{pc}}\Big|_\Gamma$ for 3 different St

Figure: 1D solidification case

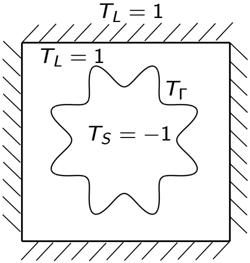
Initial position off equilibrium

In both phases: $\dfrac{\partial T}{\partial t} = D\Delta T$ $\qquad$ $\mathbf{v_{pc}}|_\Gamma \approx A \times t^{-0.5}$

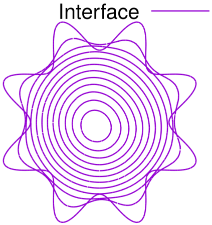
$$\lambda_L = \lambda_S = 1 \ ; \ D = 1; St = \frac{C_p \Delta T}{L_H}$$

2D star-shaped ice particle melting

- Interface $\Gamma(r,\theta)$: $r\,(1 + 0.2 * cos(8\theta)) - 1 = 0$
- Gibbs-Thomson: $T_\Gamma = T_m - \epsilon_\kappa * \kappa - \epsilon_v * v_{pc}$



(a) test



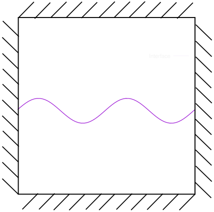
(b) Interface at different times during calculation

Figure: 2D star-shaped ice particle

Temperature field

Perspectives

- In-depth analysis of simple cases (linear stability)
- Dendrite growth
- Solidification with convection



Code available at : http://basilisk.fr/sandbox/alimare/

Thank you for your attention

Known Issue

Stencil for interpolation can be too small, leading to a decrease in the order of accuracy of the method.
⇒ Can be circumvented by using Adaptive Mesh Refinement.

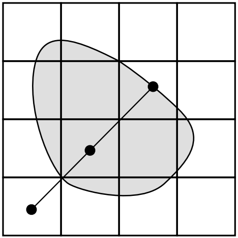


Figure: Embedded zone is too small for interpolation

Gaussian bump

Same case with an initial Gaussian interface
Case with both melting and solidification due to the Gibbs-Thomson relation.
Temperature field

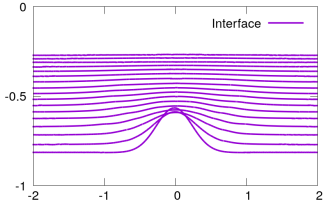


Figure: Interface at different times during the simulation

Hybrid Embedded boundaries/*level-set* method

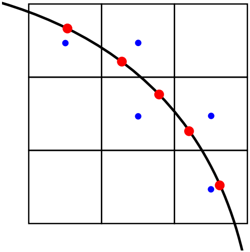

(a)

Figure: Initial condition for the v_{pc} reconstruction

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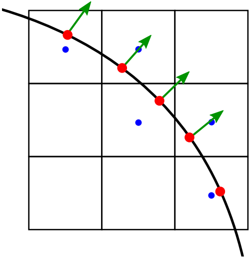

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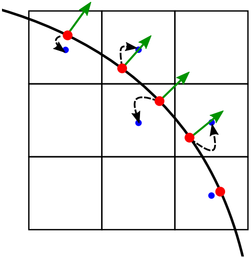

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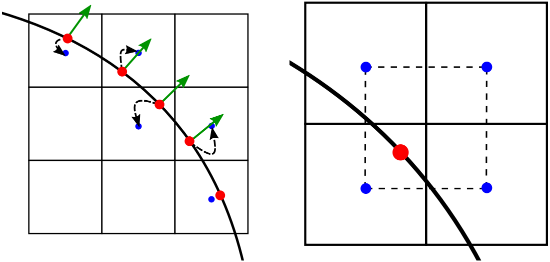

(a) (b) Local bilinear interpolation

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