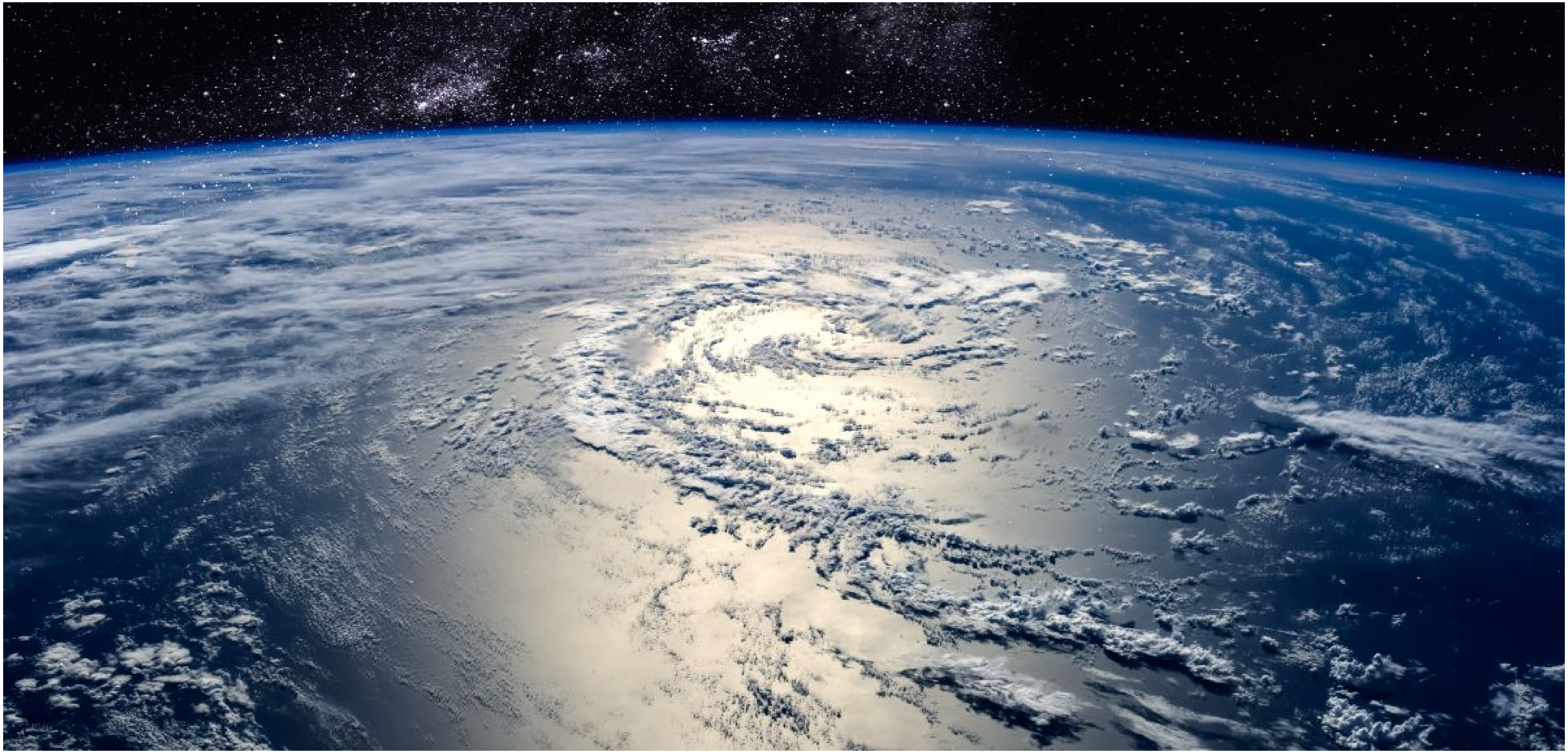
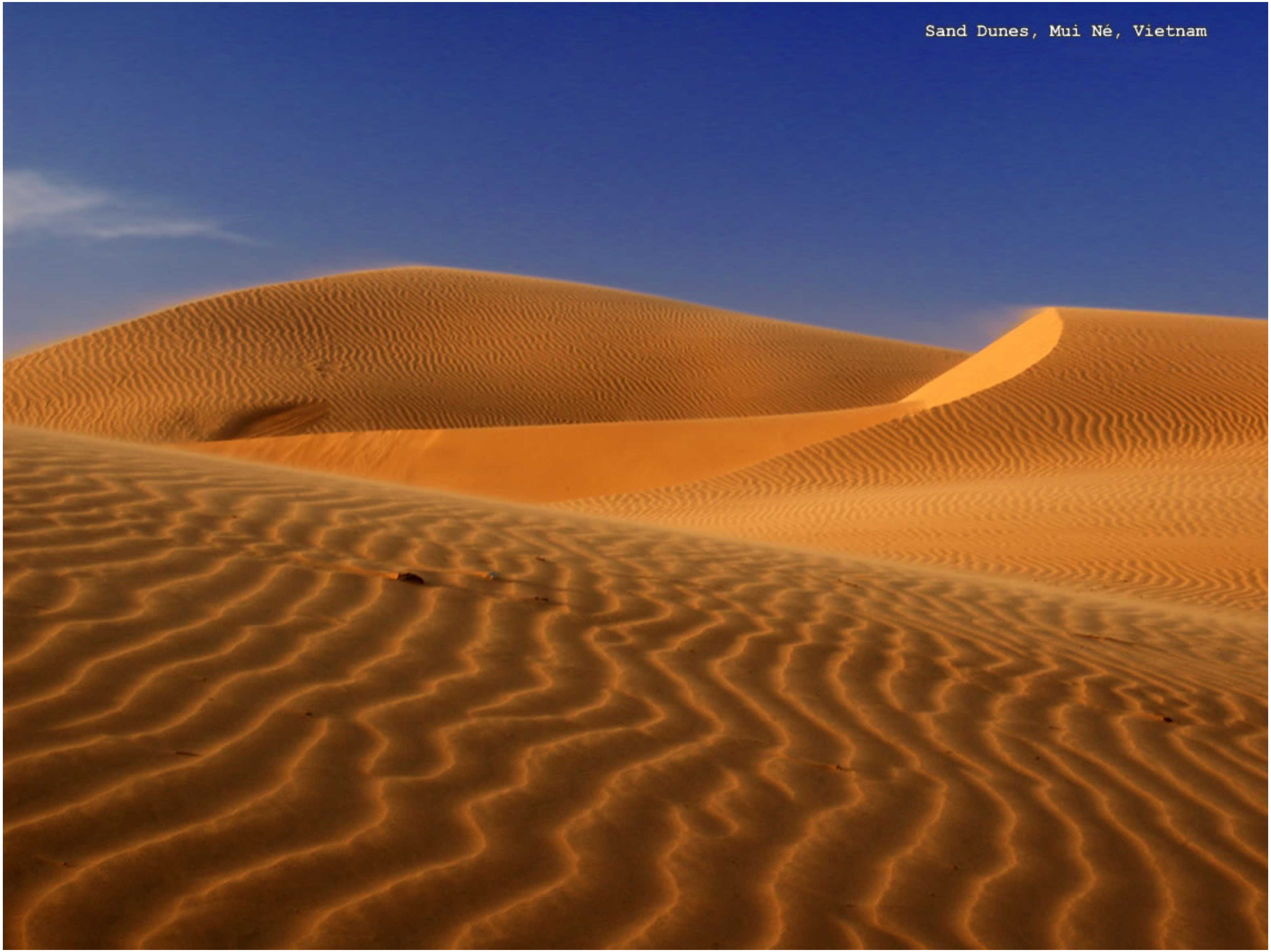
A multilayer model for non-hydrostatic multiscale free-surface flows

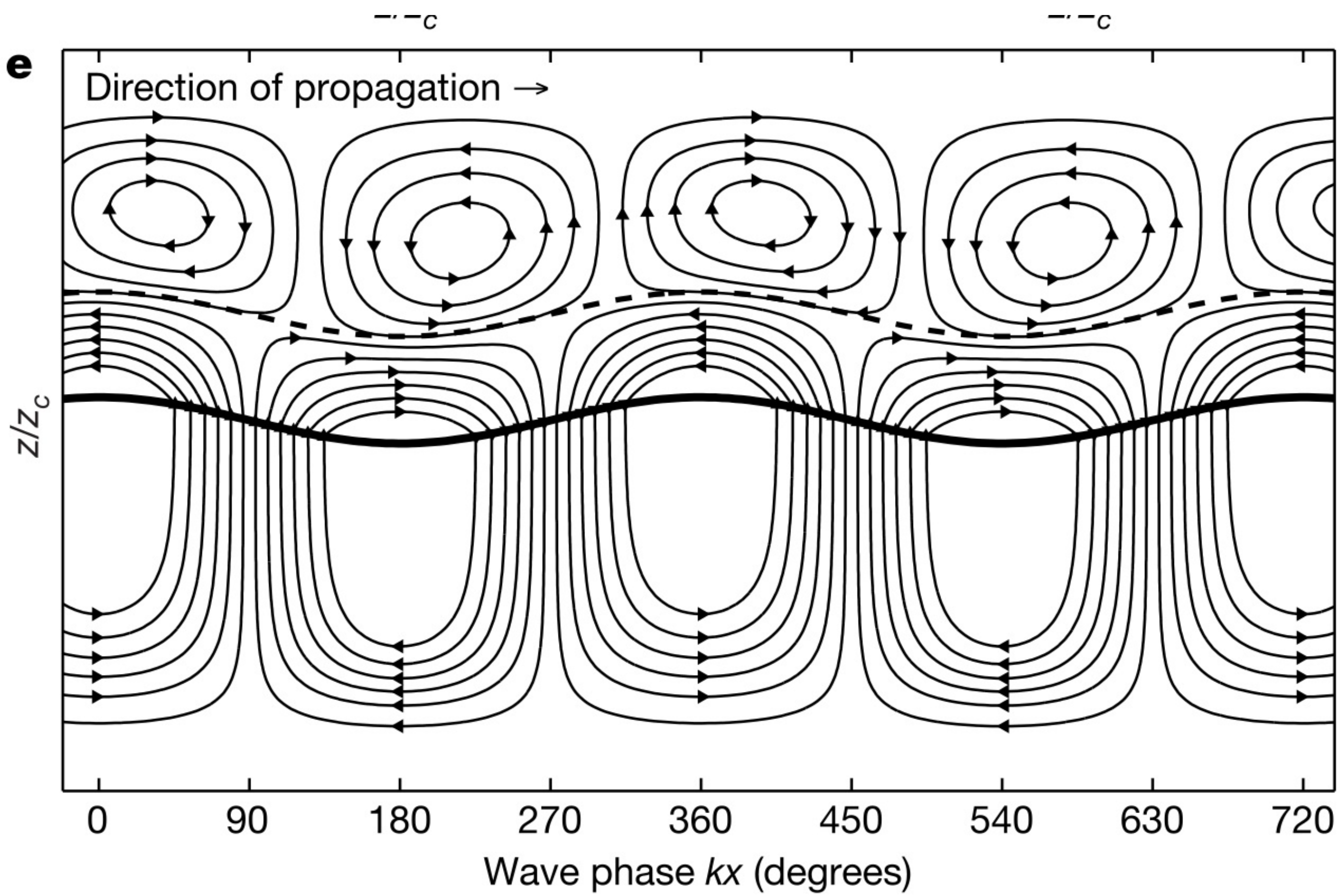
Stéphane Popinet

∂'Alembert

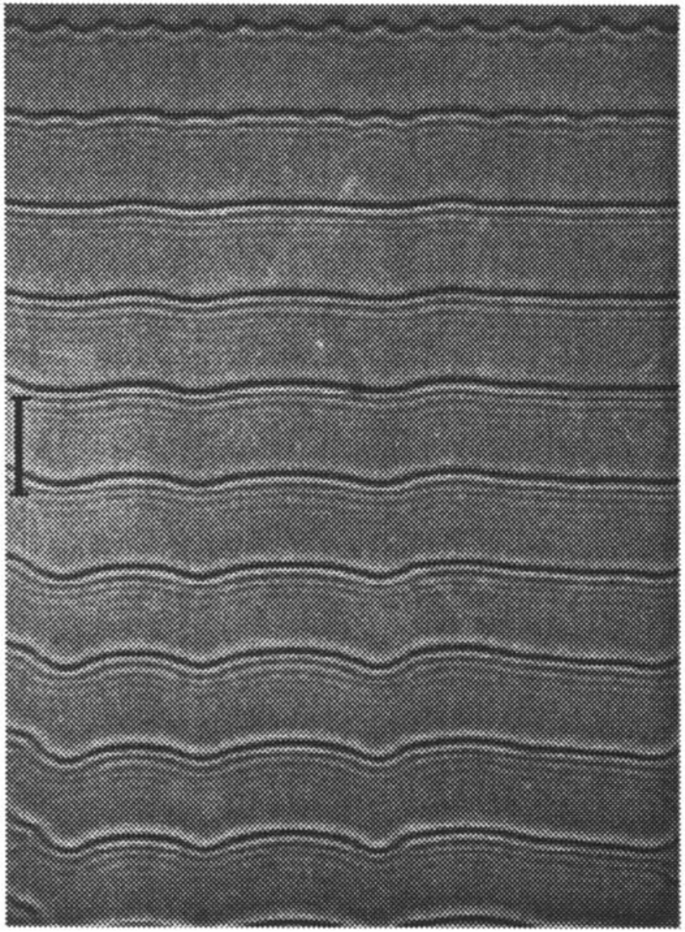
CNRS / Sorbonne Université

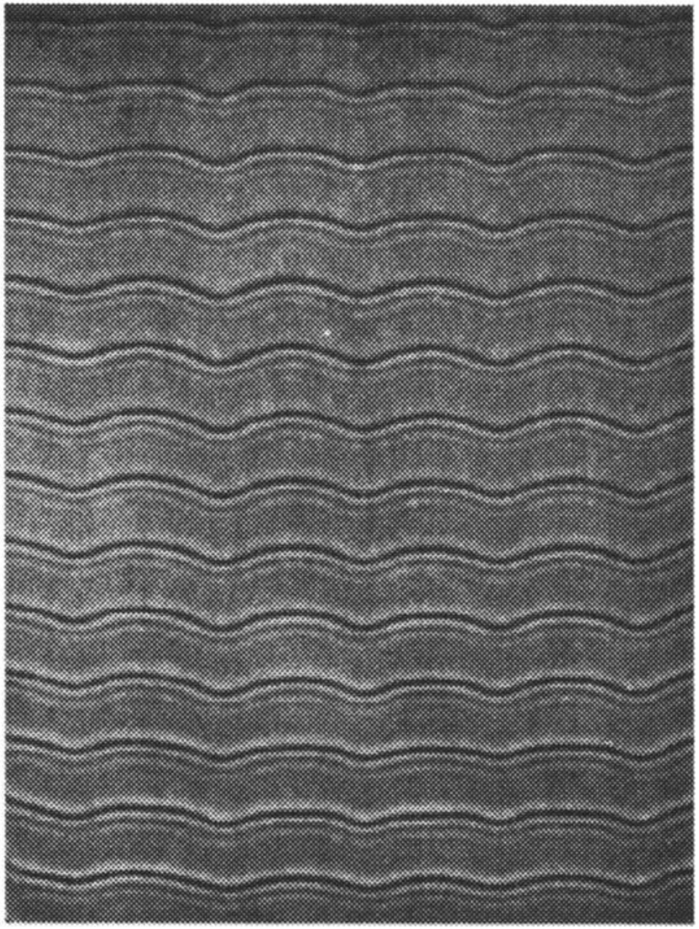



Sand Dunes, Mui Né, Vietnam



Hristov et al, *Dynamical coupling of wind and ocean waves through wave-induced air flow*, Nature, 2003.





Piotr Leonidovich Kapitza, "Wave flow of thin layers of a viscous fluid," in Collected papers of P.L. Kapitza, D. Ter Haar, Ed., pp. 662–689. Pergamon, 1948.

$$\begin{aligned}\partial_t h + \nabla\cdot(h\,\mathbf{u}) &= 0,\\ \partial_t(h\,\mathbf{u}) + \nabla\cdot(h\,\mathbf{u}\,\mathbf{u}) &= -g\,h\,\nabla\eta,\\ \eta &= z_b + h\end{aligned}$$

Adhémar Jean-Claude Barré de Saint-Venant, 1871.

- \+ Simple
- \+ System of conservation laws
- \+ Fully non-linear
- \+ Numerically efficient
- − Vertical structure is entirely modelled
- − Hydrostatic, no-vertical acceleration/momentum
- Non-dispersive waves: $c = \sqrt{g\,h}$

1797–1886

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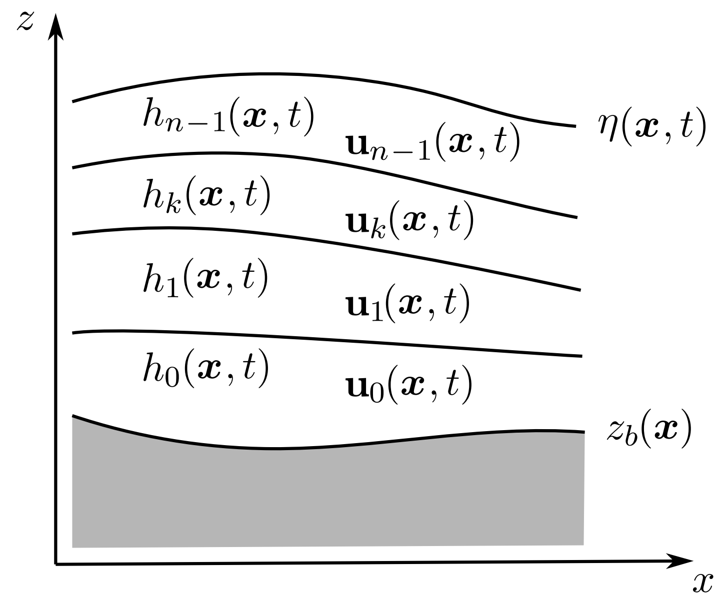

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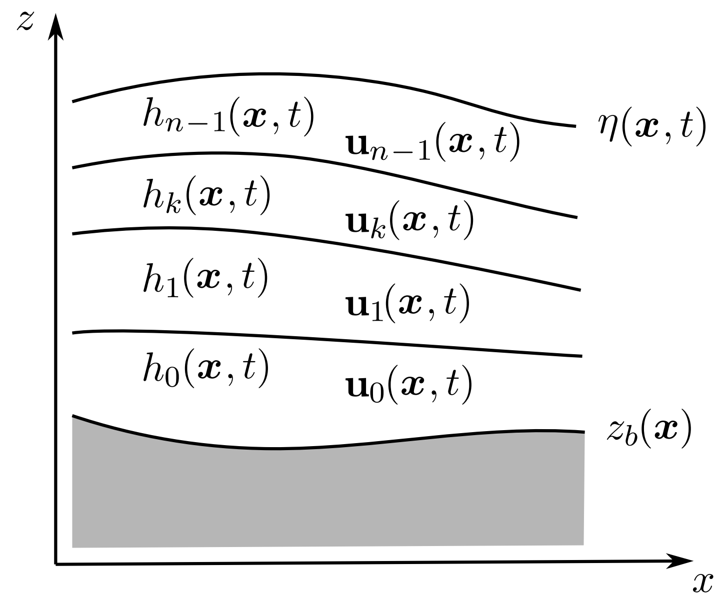

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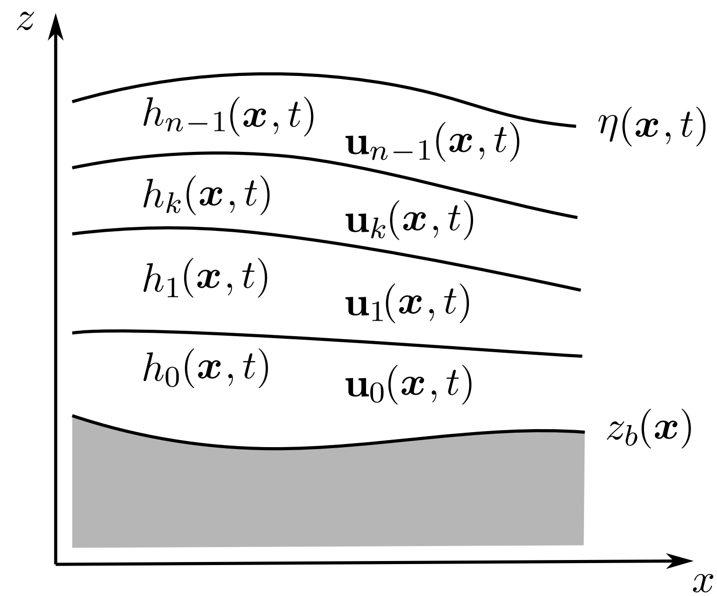

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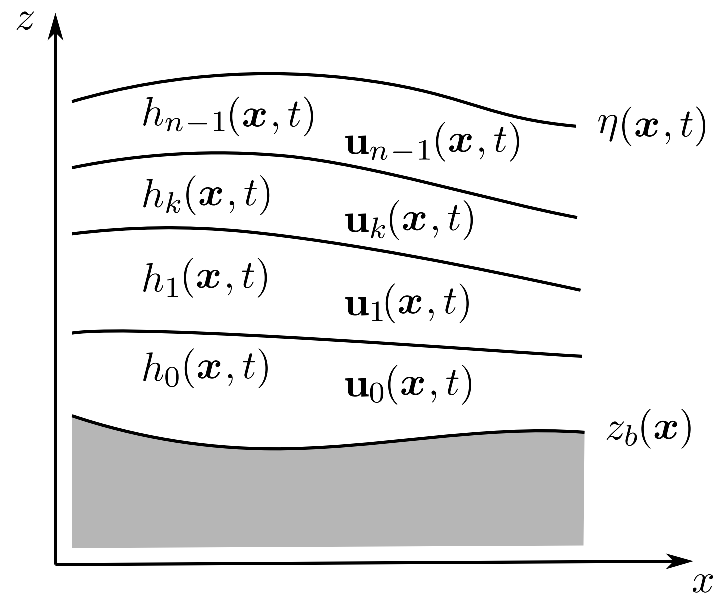

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Joseph Valentin Boussinesq, 1872 (also includes the “KdV” equation).
Many more recent variants: Serre 1953, Peregrine 1968, Green-Naghdi 1976...

- \+ Non-hydrostatic, dispersive waves: $c^2 = g\, h\, \frac{1 + k^2 h^2/6}{1 + k^2 h^2/2}$
- \+ Single layer
- − Complex asymptotic terms
- − High-order derivatives
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1842–1929

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Toward non-hydrostaticity: Boussinesq-style systems

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Leonhard Euler, 1757 (without the free surface).

\+ Complete/consistent system
\- Complex boundary condition
\- Link with Saint-Venant?
\- How to represent χ? (VOF, Levelset, Lagrangian etc.)
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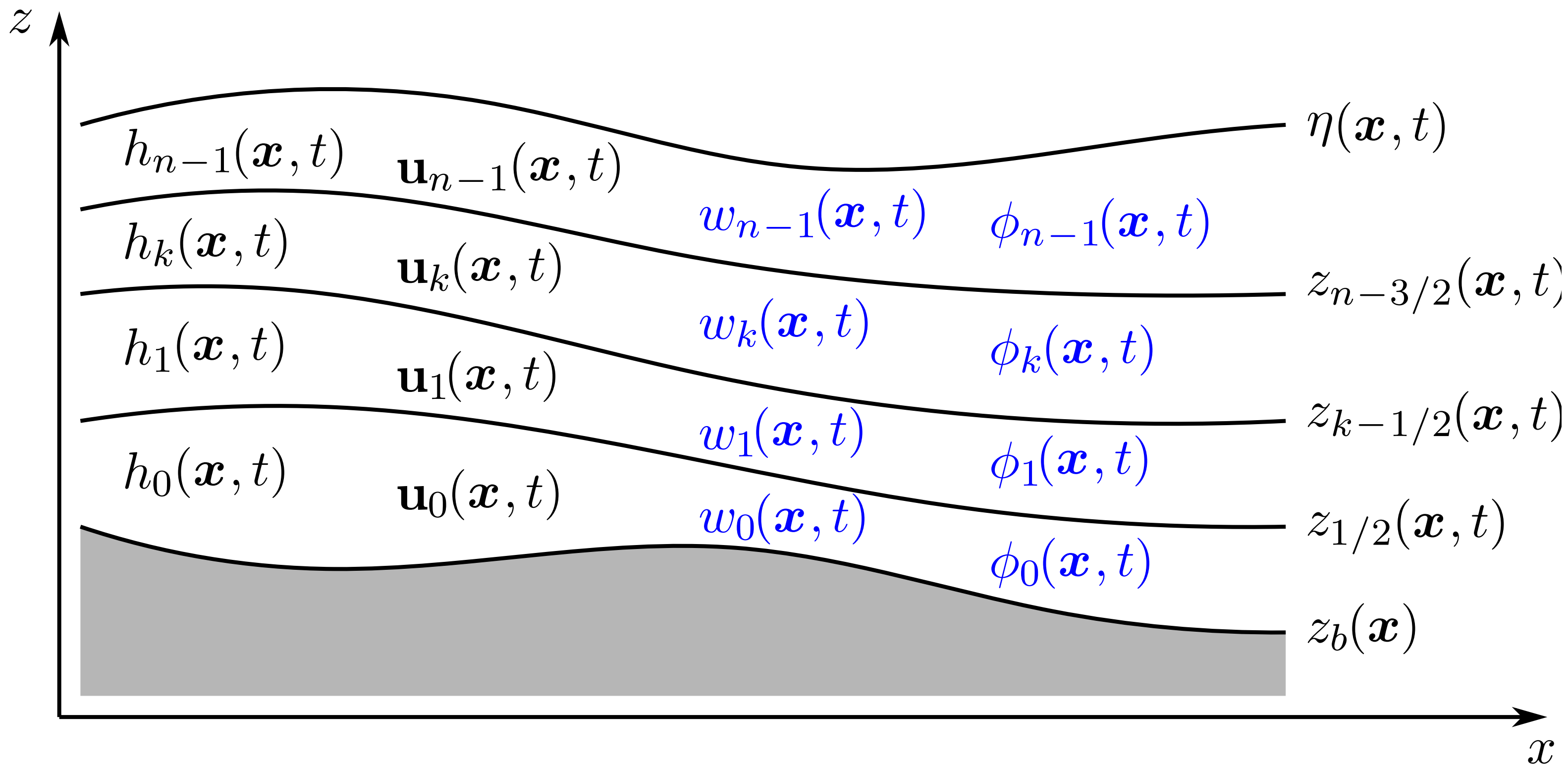

$\phi = \frac{p_{\mathrm{nh}}}{\rho}$ with p_{nh} the non-hydrostatic pressure.

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[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}$$

- \+ Consistent discretisation of the full 3D Euler equations
- \+ Clearly connected with the simpler hydrostatic system
- \+ System of (discrete) conservation laws
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- − Too many unknowns!

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\partial_t (h\,w)_k + \nabla\cdot(h\,w\,\mathbf{u})_k &= -[\phi]_k,\\
\nabla\cdot(h\,\mathbf{u})_k + [w - \mathbf{u}\cdot\nabla z]_k &= 0,\\
[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}
$$

- \+ Consistent discretisation of the full 3D Euler equations
- \+ Clearly connected with the simpler hydrostatic system
- \+ System of (discrete) conservation laws
- \+ Numerically efficient
- +− Vertically Lagrangian
- − Too many unknowns!

$$
\begin{aligned}
\partial_t h_k + \boldsymbol{\nabla} \cdot (h\,\mathbf{u})_k &= 0,\\
\partial_t (h\,\mathbf{u})_k + \boldsymbol{\nabla} \cdot (h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}(h\,\phi)_k + [\phi\,\boldsymbol{\nabla} z]_k,\\
\partial_t (h\,w)_k + \boldsymbol{\nabla} \cdot (h\,w\,\mathbf{u})_k &= -[\phi]_k,\\
\boldsymbol{\nabla} \cdot (h\,\mathbf{u})_k + [w - \mathbf{u}\cdot\boldsymbol{\nabla} z]_k &= 0,\\
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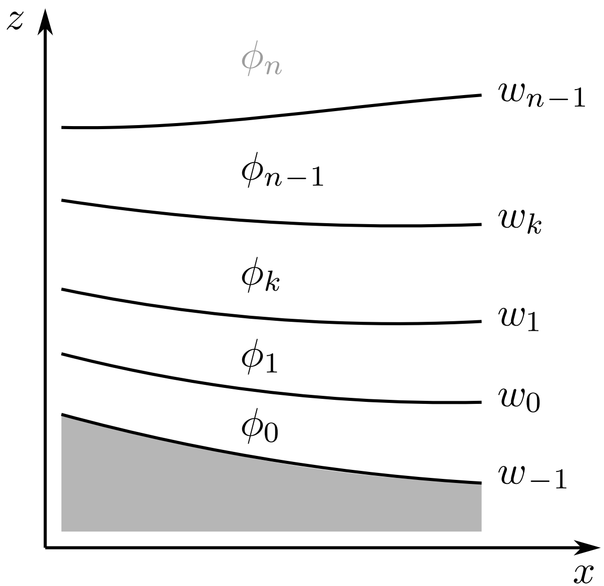
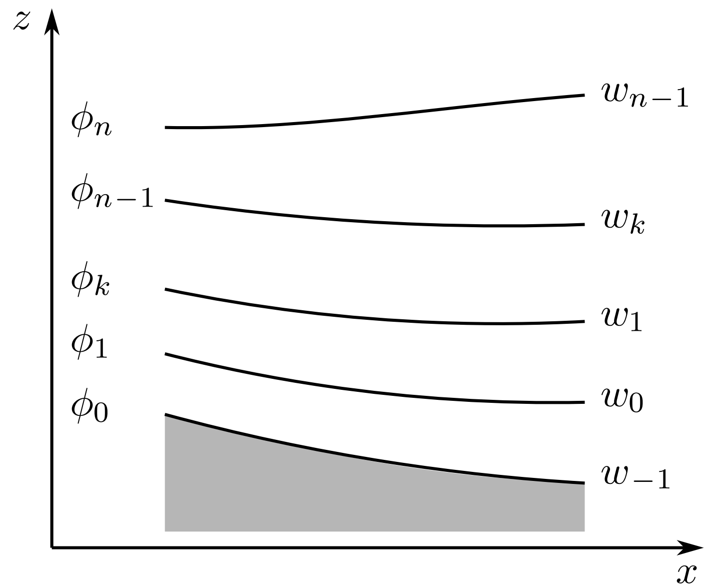
Examples of possible vertical discretisations

1917–2008



Lorenz grid



Keller grid (box scheme)

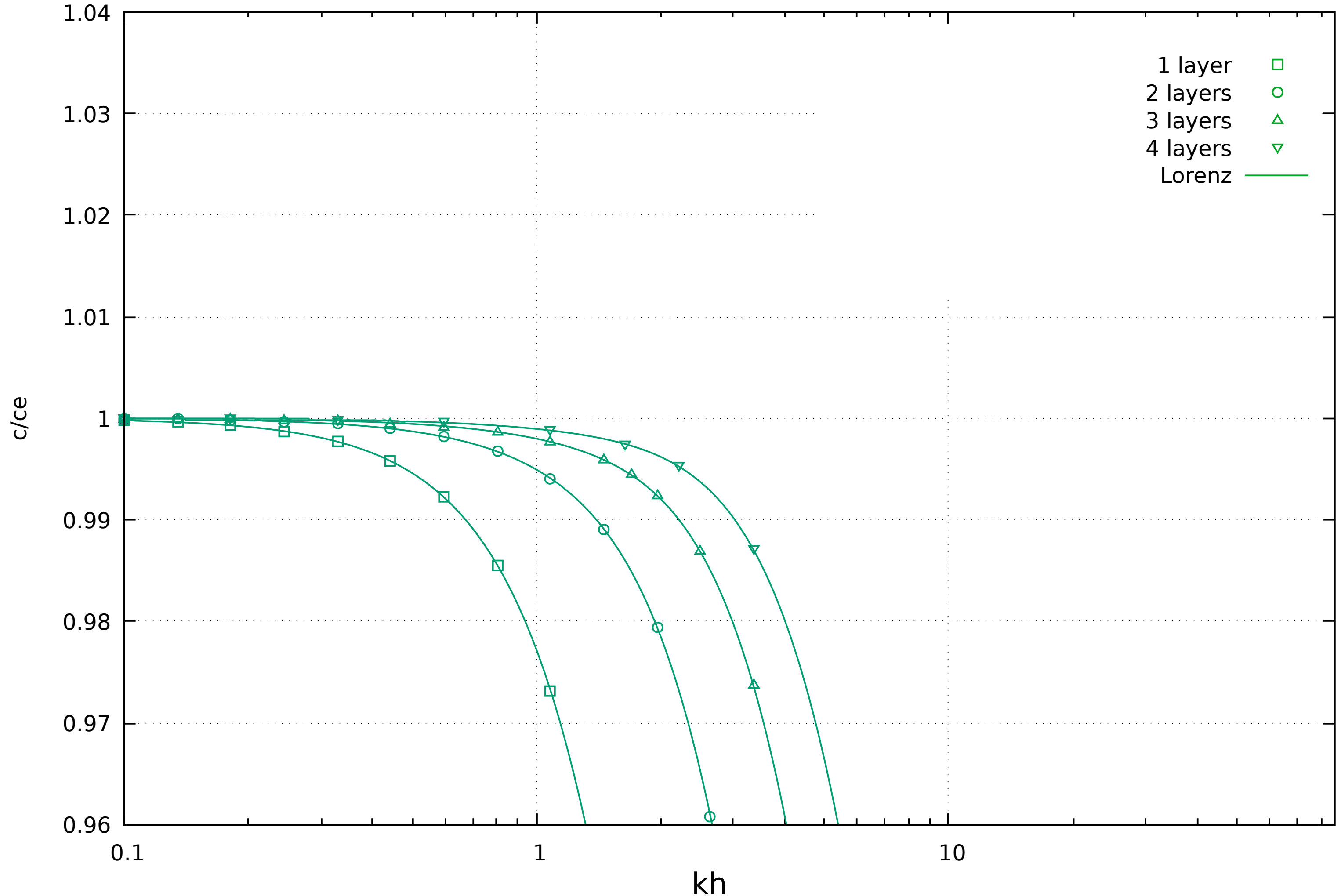
Edward Norton Lorenz, *Energy and numerical weather prediction*, 1960.

Which one is best? $\rightarrow$ choose the best dispersion relation based on the *linearised perturbation system* ($h_k = \bar{h}_k + h'_k\, e^{i(\hat{k}x - \omega t)}$, $u_k = u'_k\, e^{i(\hat{k}x-\omega t)}$, etc.)

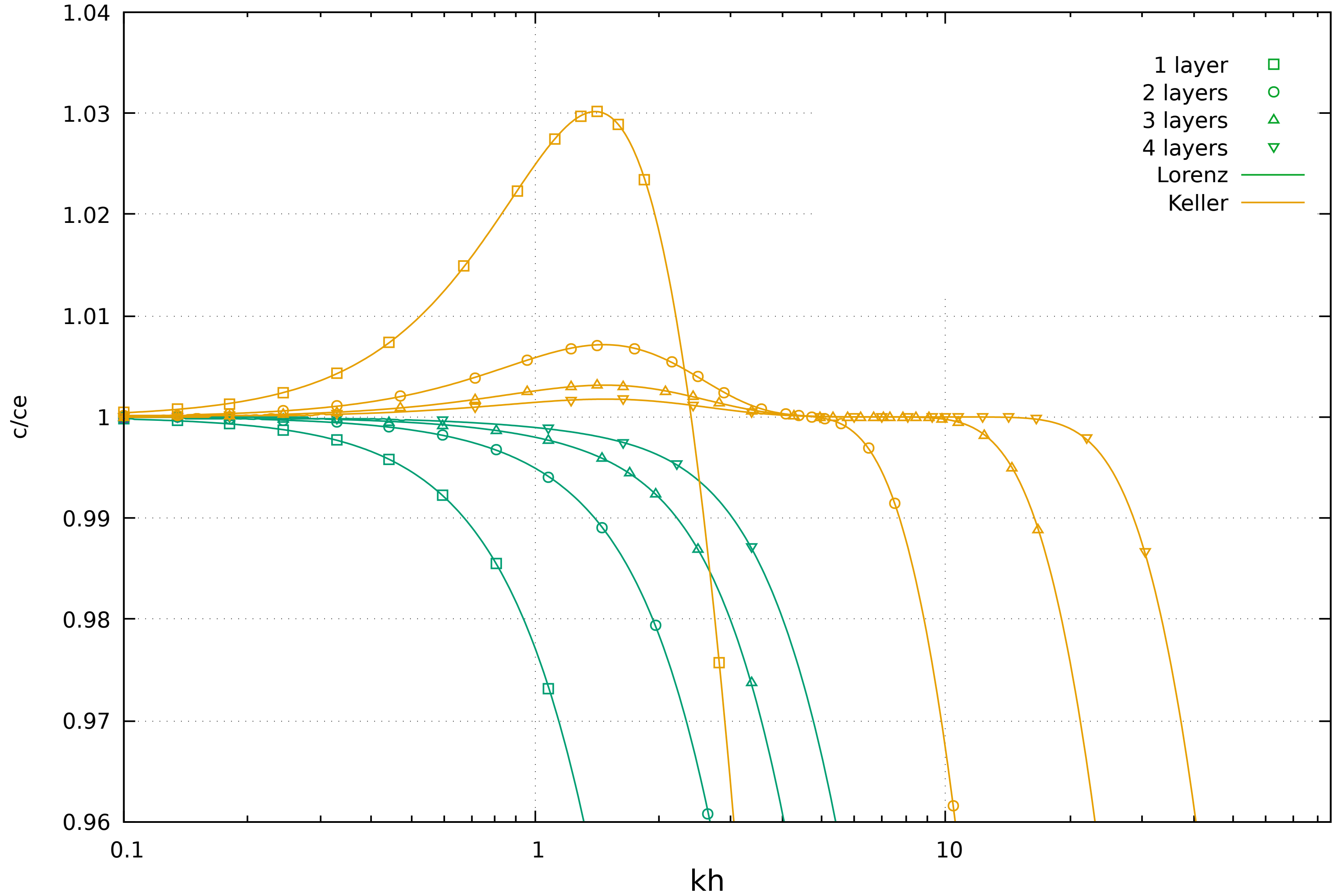
$$
\begin{aligned}
-\omega\, h'_k + \bar{h}_k\, u'_k\, \hat{k} &= 0, \\
-\bar{h}_k\, u'_k\, \omega &= -g\, \hat{k}\, \bar{h}_k \sum h'_k - \bar{h}_k\, \hat{k}\, \frac{\phi'_{k+1} + \phi'_k}{2}, \\
-\bar{h}_k\, \frac{w'_k + w'_{k-1}}{2}\, \omega\, i &= \phi'_k - \phi'_{k+1}, \\
\bar{h}_k\, u'_k\, \hat{k}\, i + w'_k - w'_{k-1} &= 0,
\end{aligned}
$$

Computing the determinant then gives (for two layers)

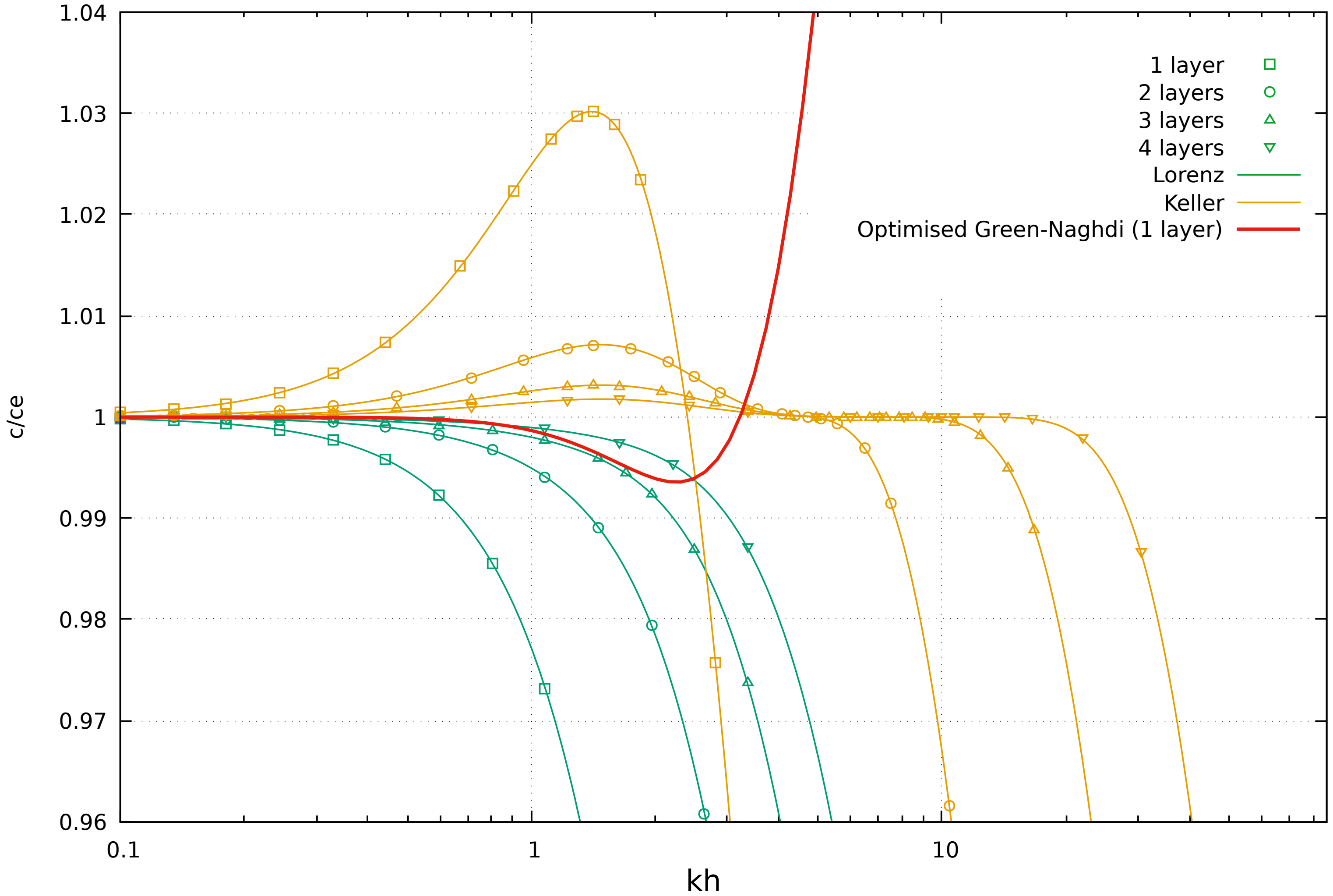
$$
\omega_2^2(\hat{k}) = 8\, g\, \frac{h_0\, h_1^2\, \hat{k}^4 + (h_1 + h_0)\, \hat{k}^2}{3\, h_0^2\, h_1^2\, \hat{k}^4 + 3\,(3\, h_0^2 + h_0\, h_1 + h_1^2)\, \hat{k}^2 + 8}
$$



$$c_e^2 = \frac{g}{k}\tanh(k\,h)$$

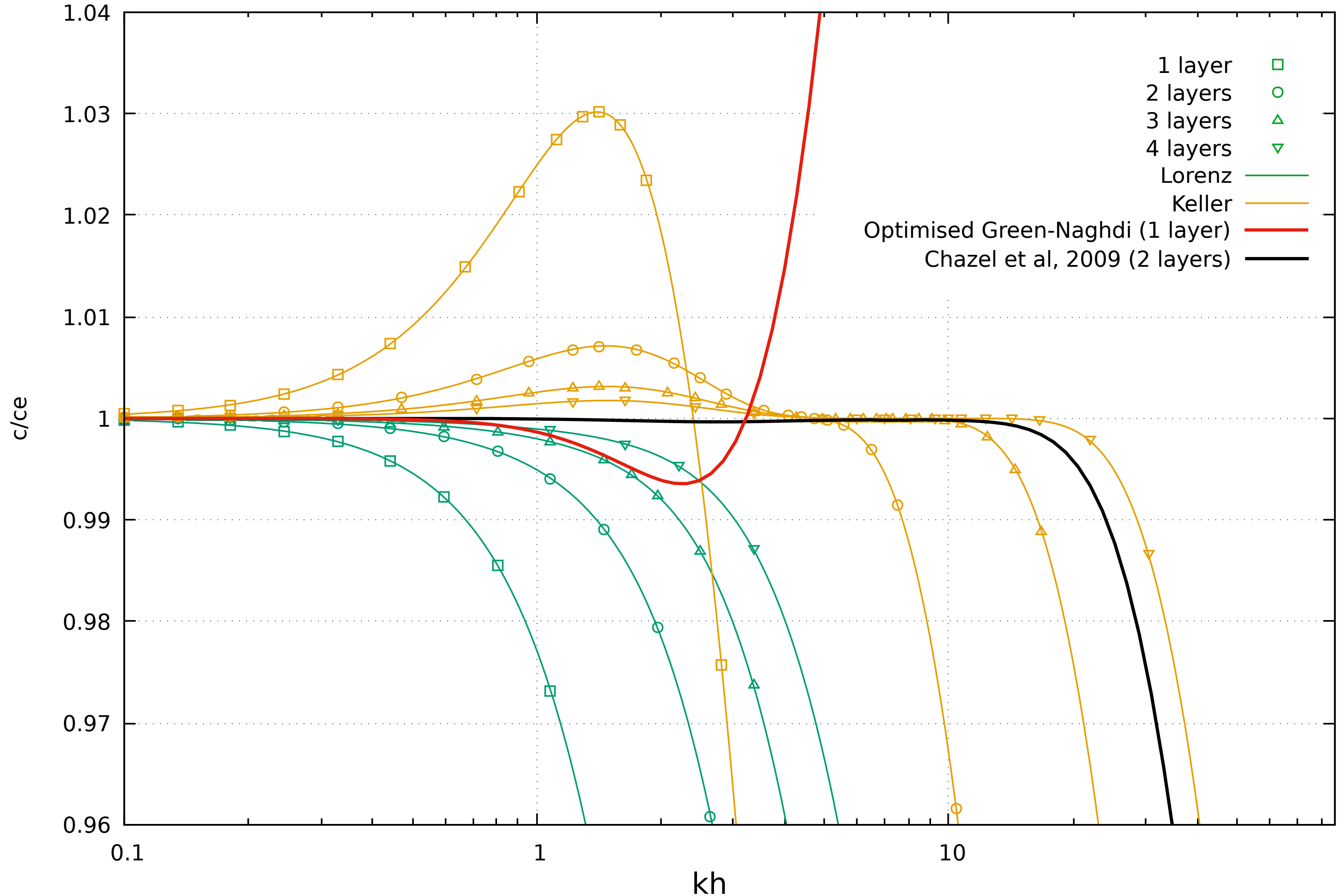


$$c_e^2 = \frac{g}{k}\tanh(k\,h)$$

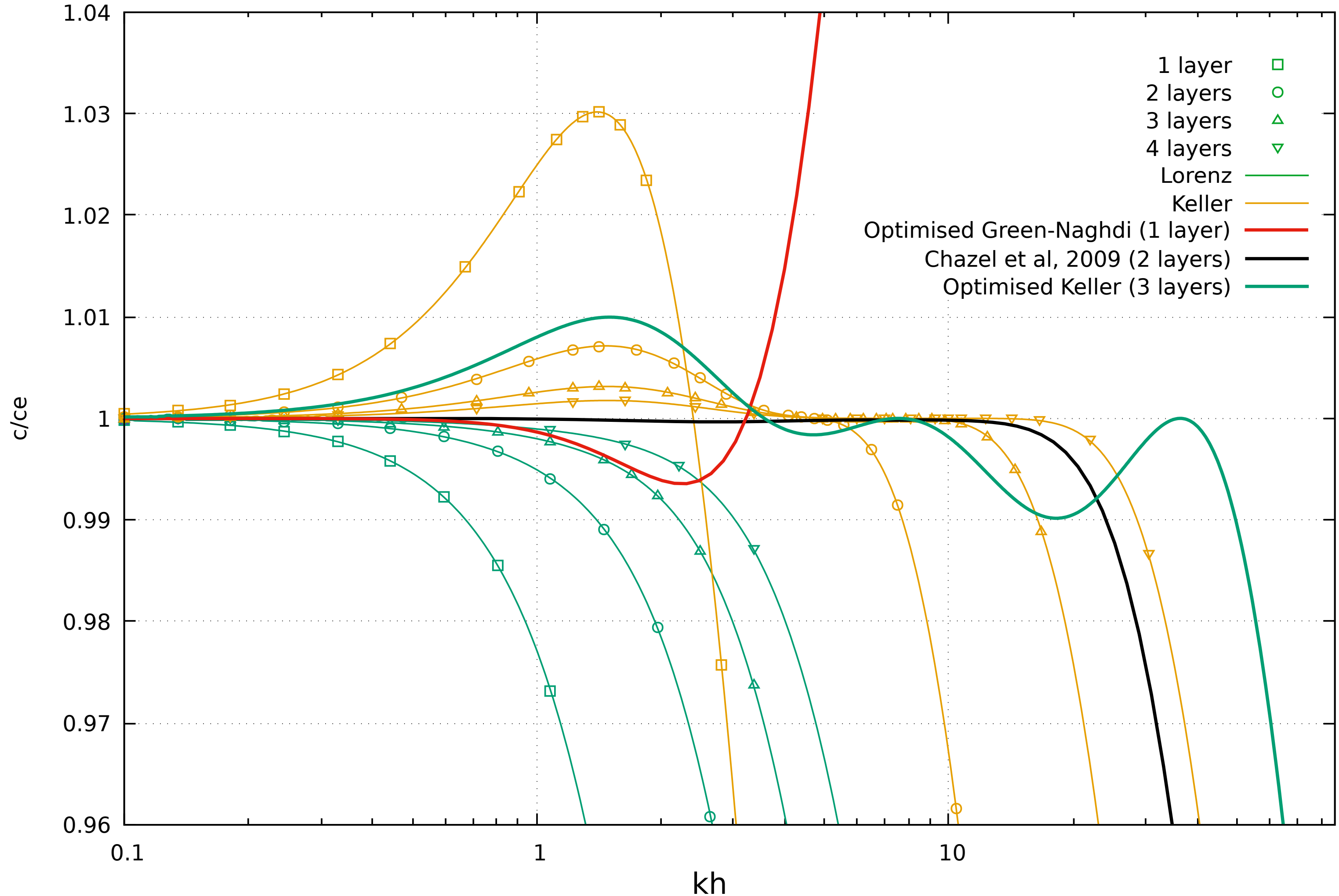


$$c_e^2 = \frac{g}{k} \tanh(k\,h)$$

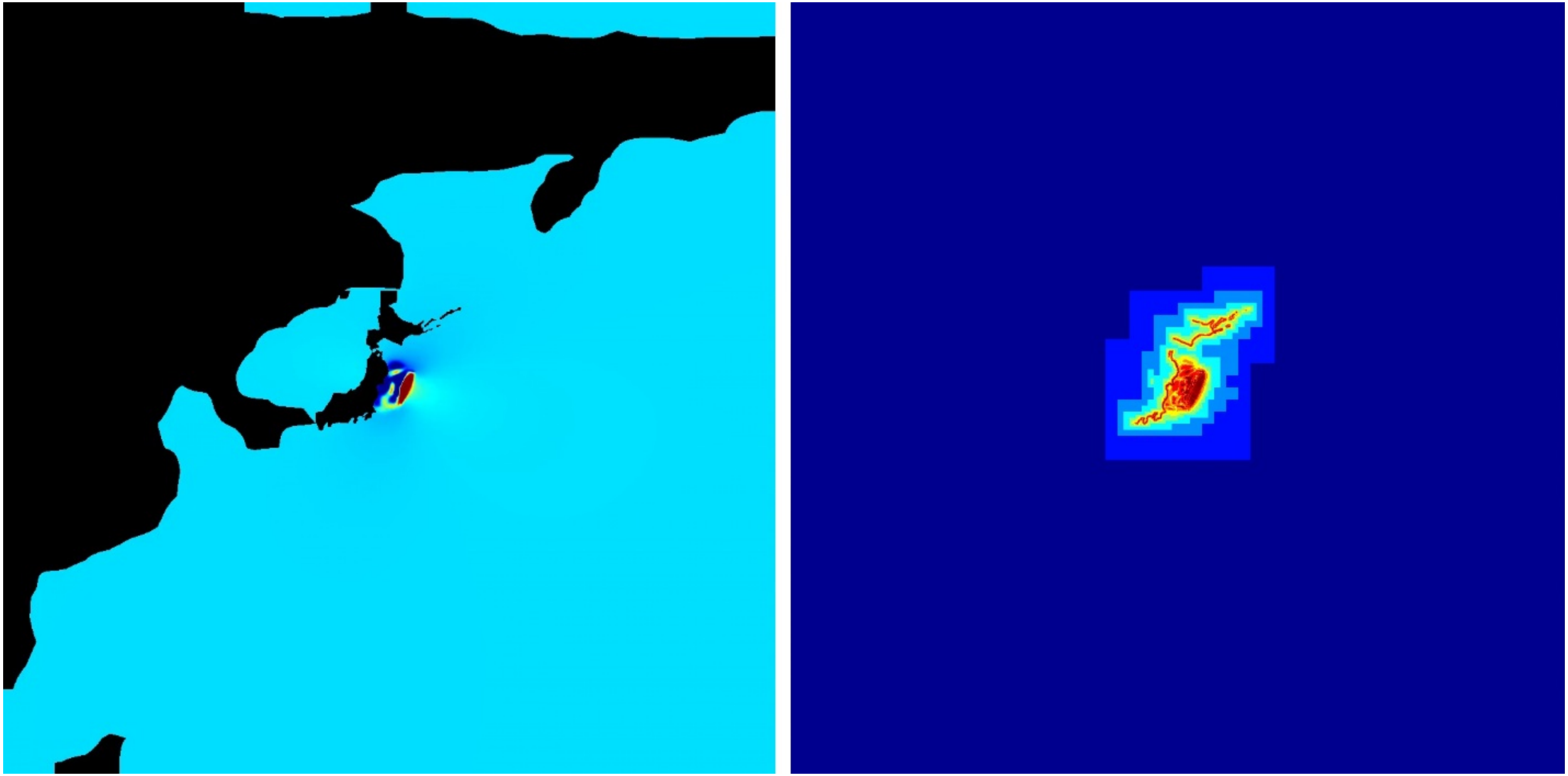
Dispersion relations



$$c_e^2 = \frac{g}{k}\tanh(k\,h)$$

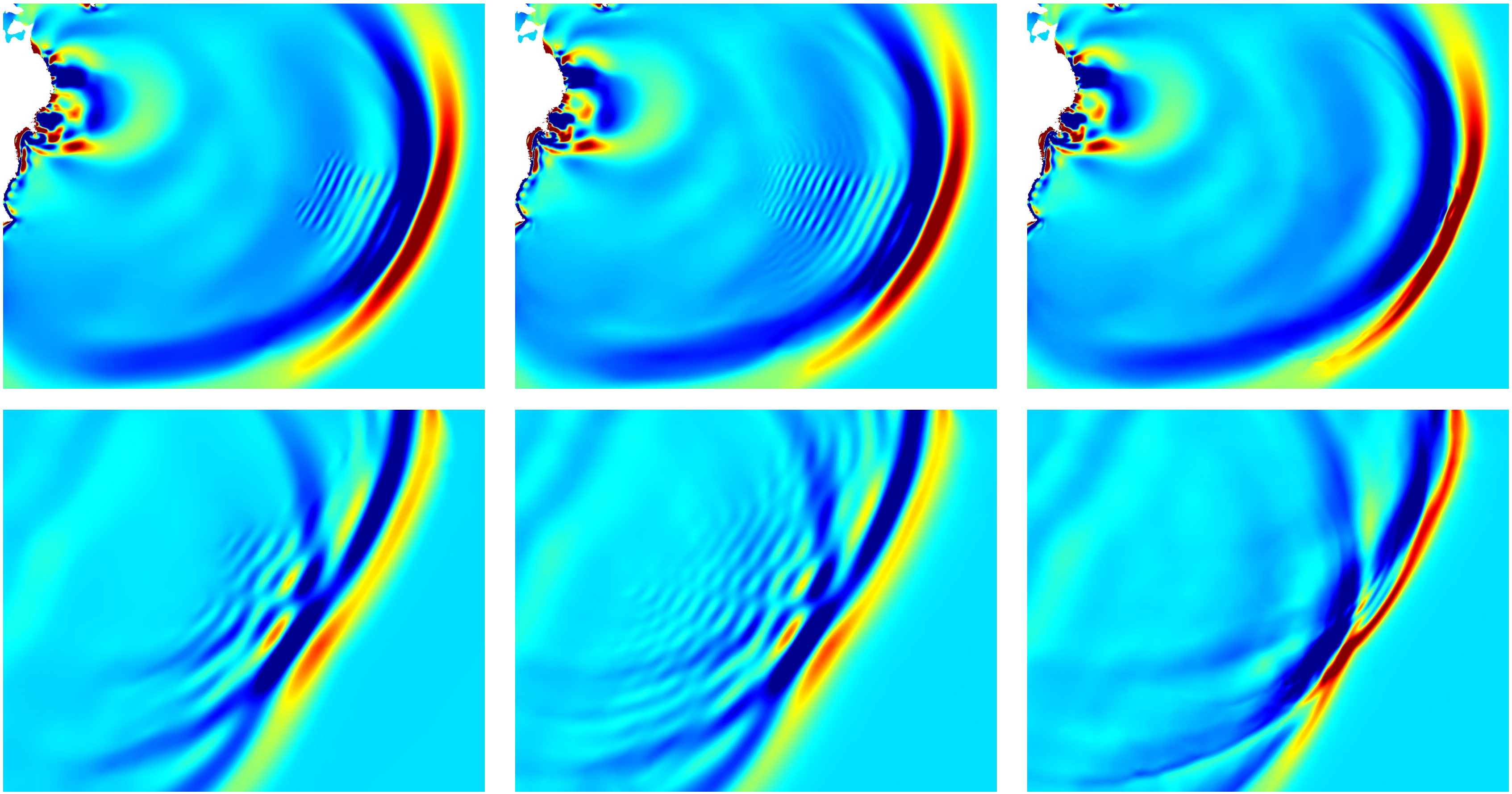


$$c_e^2 = \frac{g}{k} \tanh(k\,h)$$



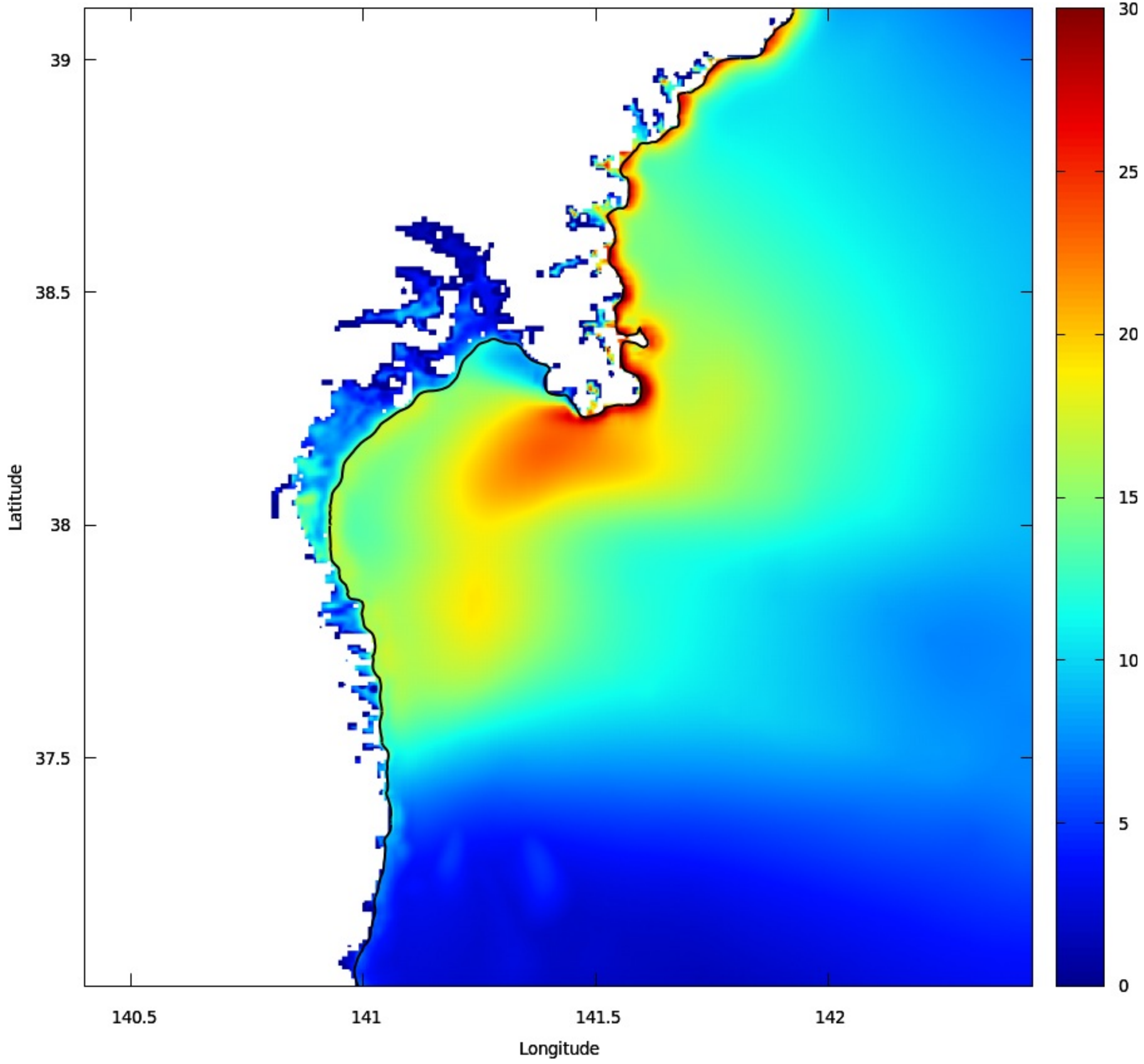
6000×8000 km, spatial resolution: 1–250 km

Serre–Green–Naghdi | One-layer non-hydrostatic | One-layer hydrostatic



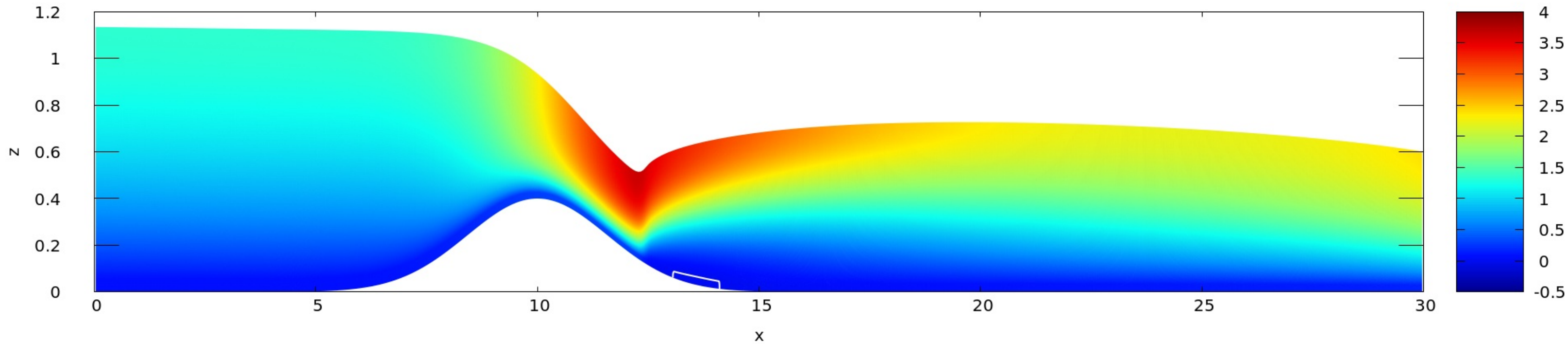
CPU runtime: 7h30 | CPU runtime: 4h15 | CPU runtime: 5h15

Detail of flooding

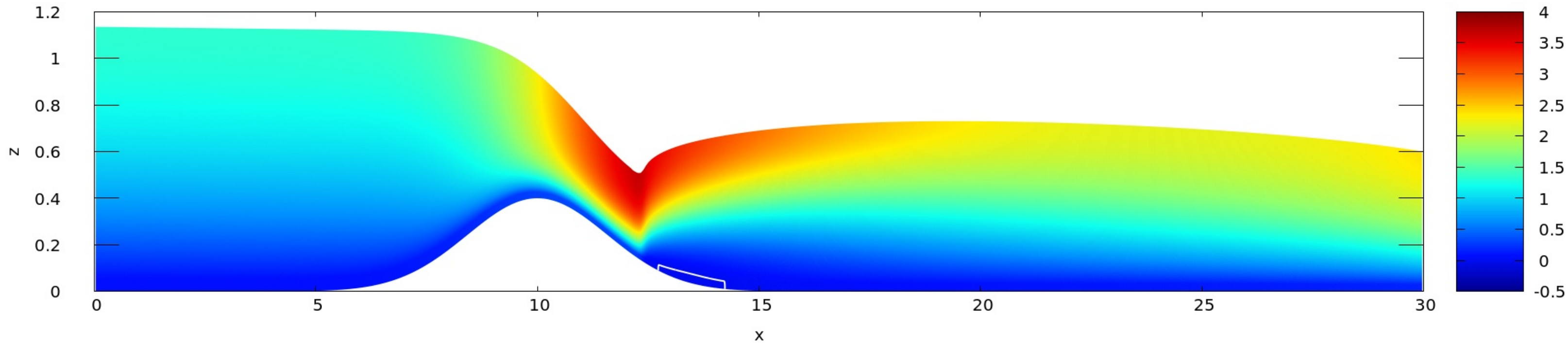


Sendai plain: 140×200 km

Horizontal velocity

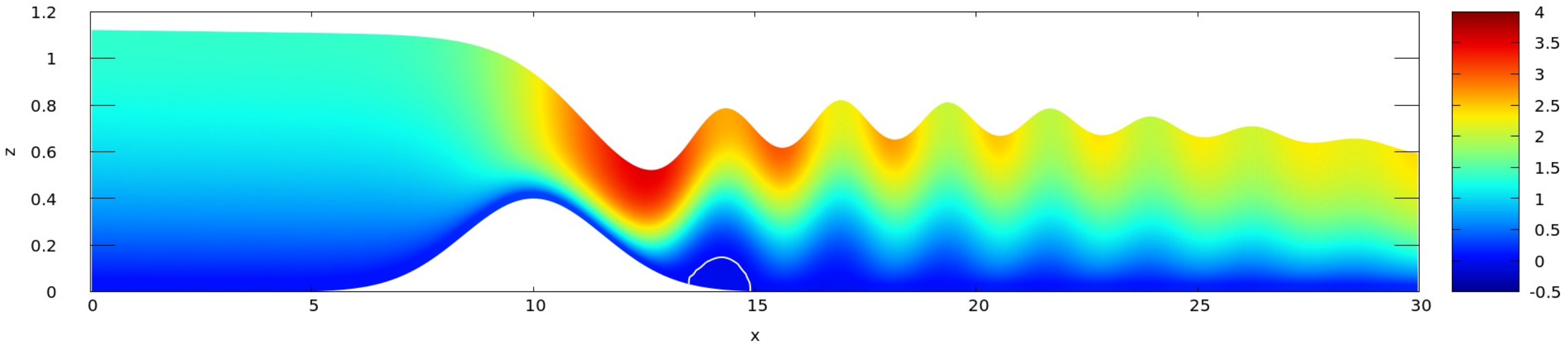


Multilayer hydrostatic

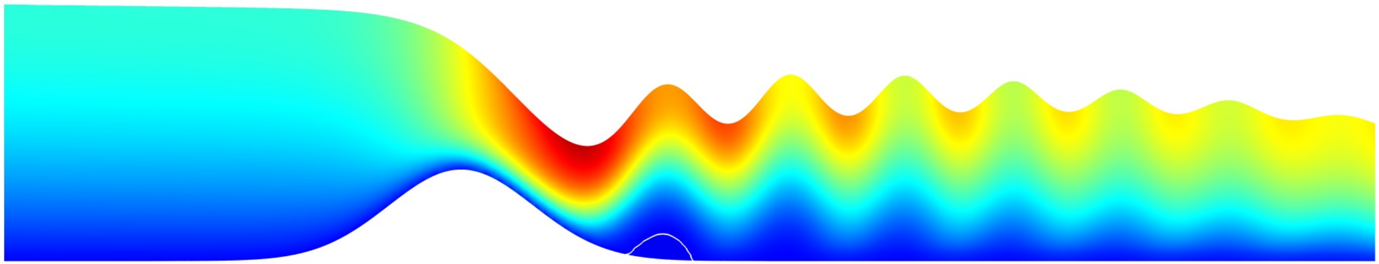


Riemann solver

Horizontal velocity

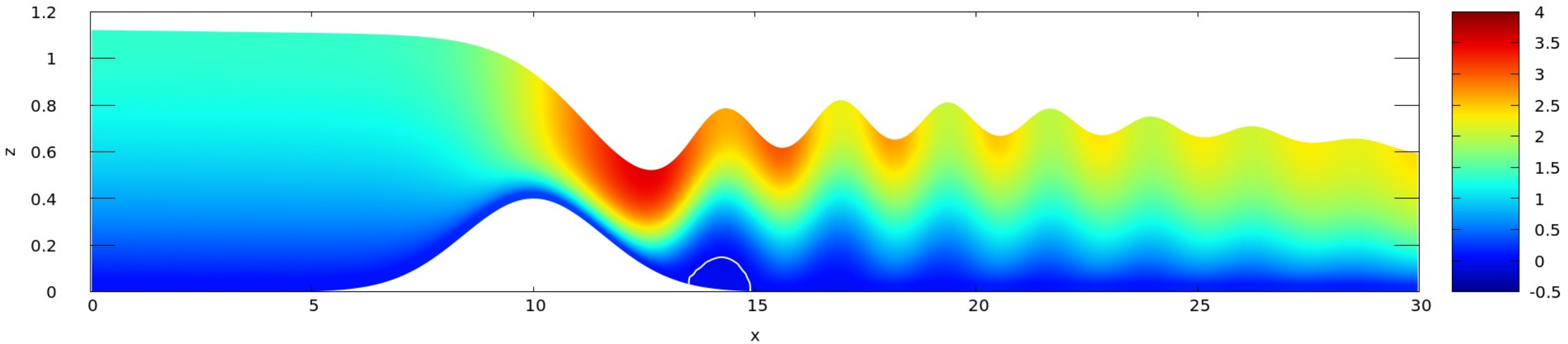


Multilayer

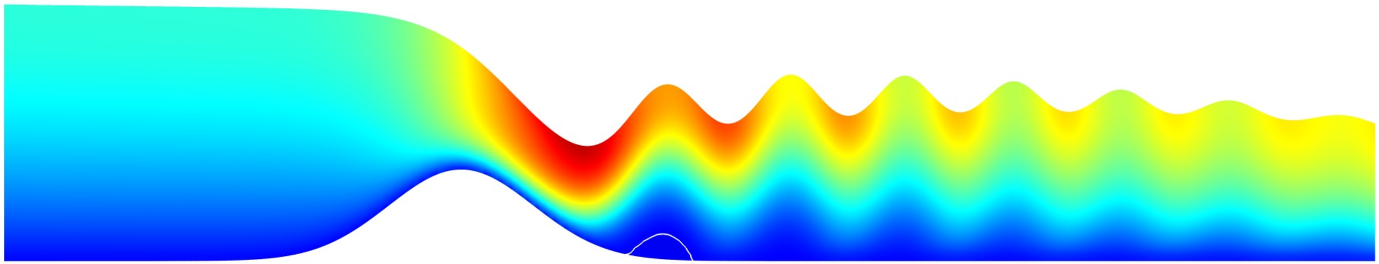


Navier–Stokes VOF

Horizontal velocity

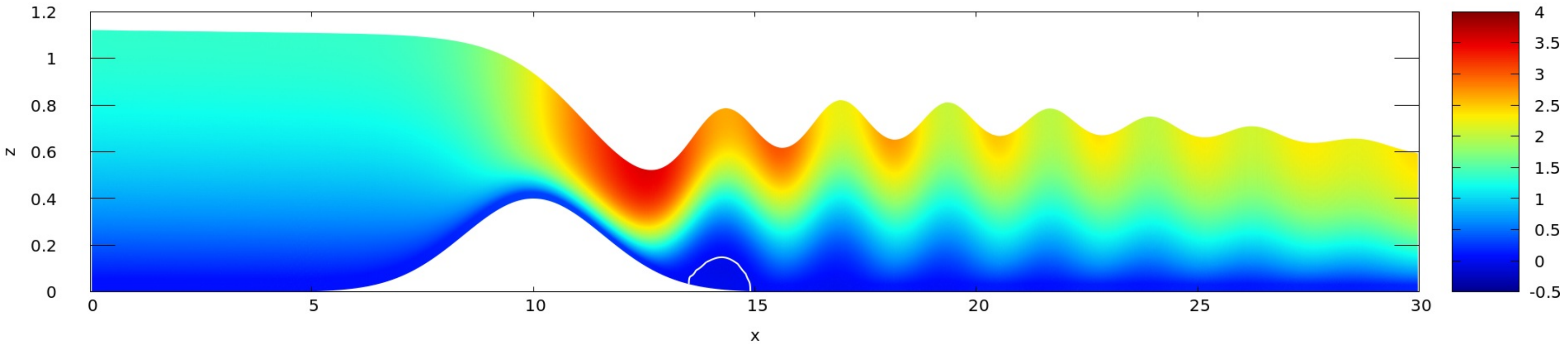


Multilayer (runtime: 3 min)

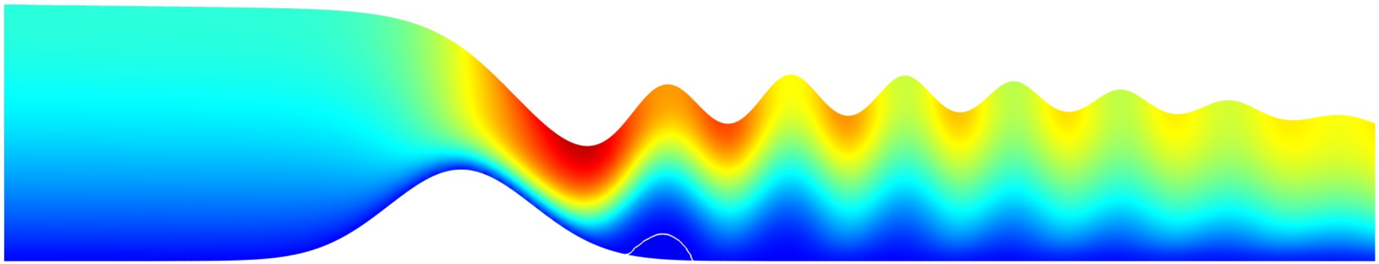


Navier–Stokes VOF

Horizontal velocity

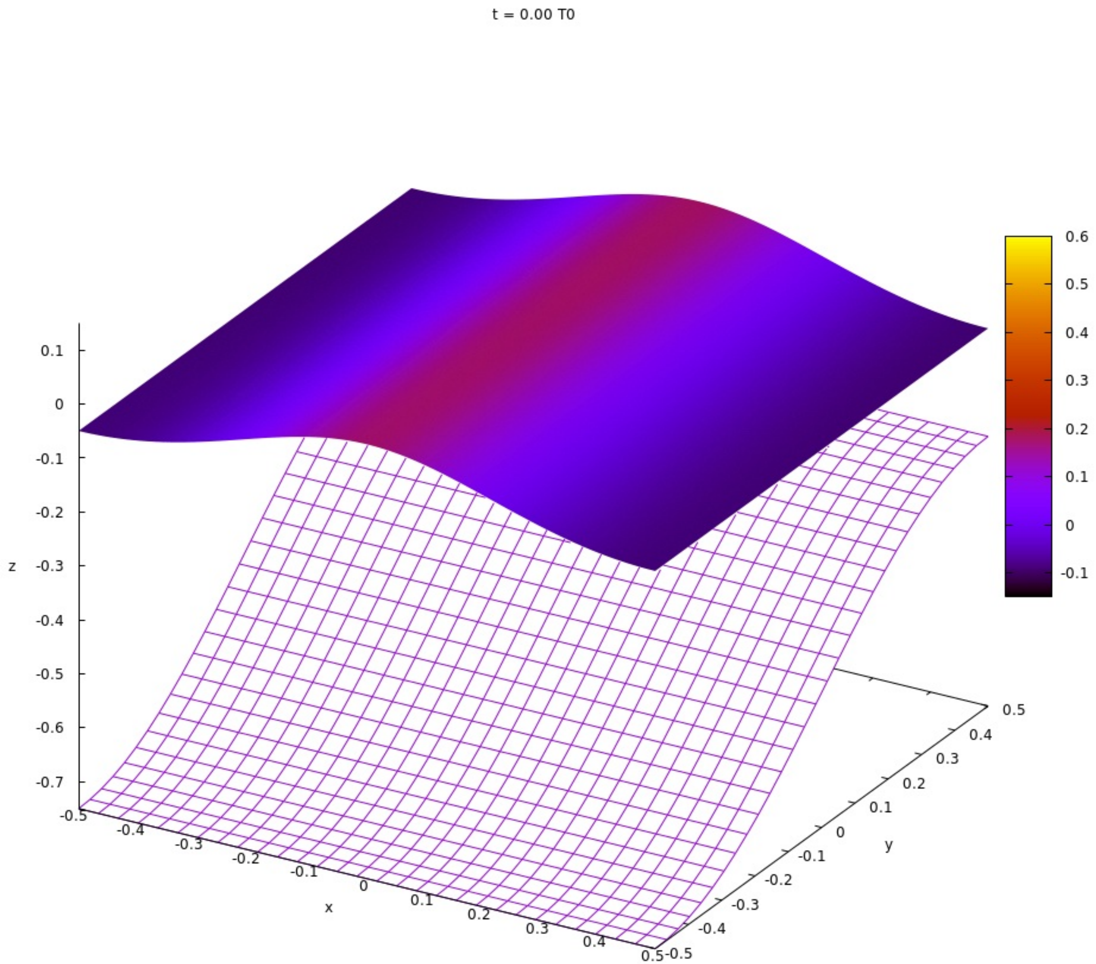


Multilayer (runtime: 3 min)



Navier–Stokes VOF (runtime: 1h15)

Standing “river” waves in Waimea (Hawaii)



runtime: 13 minutes on 64 cores

- A semi-discrete consistent representation of the incompressible Euler/Navier–Stokes equations with a free-surface.
- This new set of equations has a clear physical interpretation and makes a seamless link between the Euler, Saint-Venant and Boussinesq equations.
- The same model gives accurate and efficient solutions for the evolution of metre-scale to kilometer-scale waves.
- Work in progress:
  - Multimaterial flows: densities, rheologies, surface tension etc.
  - Coriolis forces / geostrophic balance
  - Applications to ocean modelling
- Also an ideal model for phase change…
- Preprint on HAL and `basilisk.fr`