Modelling fallling film flow: an adjustable formulation

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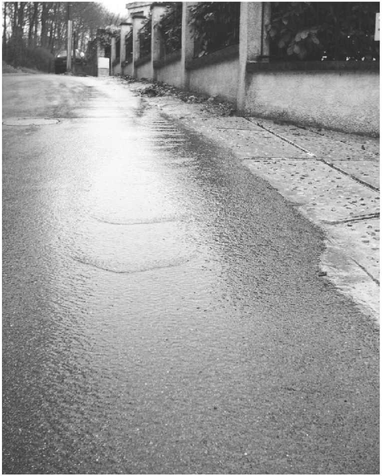
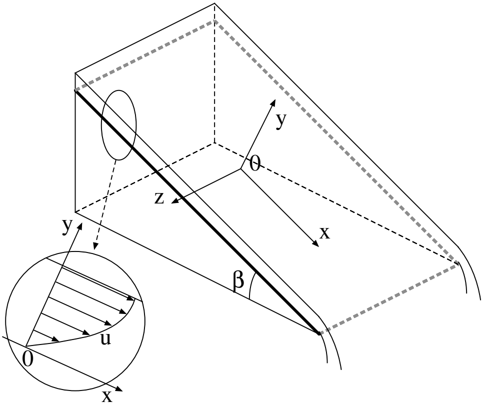
Falling films down an inclined plane



Raining days[1]

[1] Thanks to Frédéric Moisy !

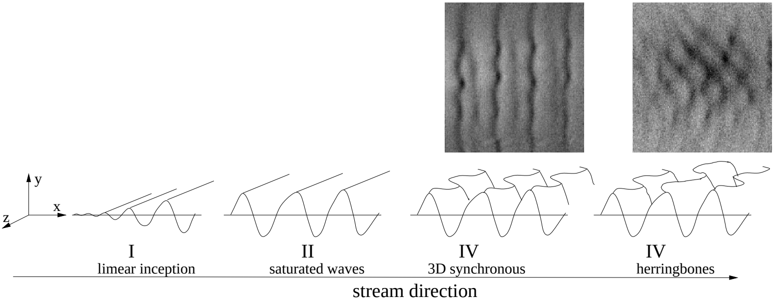
Position



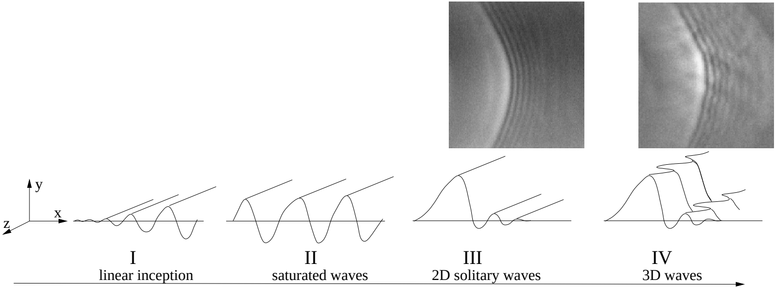
surface tension σ, viscosity μ, density ρ, gravity g
inclination angle β, inlet flow rate per wetted perimeter $\bar{q}_{\mathrm{N}}$
streamwise x, spanwise z, cross-stream y directions

A series of symmetry breakings

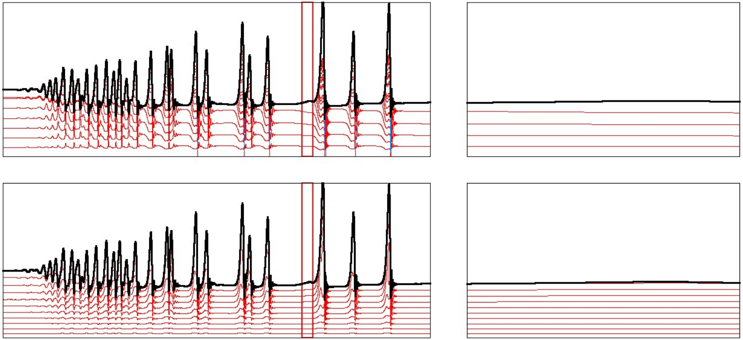
high frequency



low frequency



Noise-driven dynamics



Kapitza experiments[2] alcohol film on vertical wall $R = 6.07$, $\Gamma = 529$, length $L = 80$ cm, decelerated 8 times

[2] Kapitza & Kapitza Zh. Ekper.Teor. Fiz. 19, 105-120 (1949)

Parameters

scales based on kinematic viscosity $\nu = \mu/\rho$ and gravity :

$$l_\nu = \nu^{2/3}(g\sin\beta)^{-1/3} \quad \text{and} \quad t_\nu = \nu^{1/3}(g\sin\beta)^{-2/3}.$$

and Nusselt solution

$$u = \frac{g\sin\beta}{2} y(2\bar{h}_N - y)$$

$$\bar{u}_N = \bar{h}_N^{-1} \int_0^{\bar{h}_N} u dy = \frac{g\sin\beta}{3\nu} \bar{h}_N^2$$

$\rightsquigarrow$parameters :

- dimensionless Nusselt thickness $h_N = \bar{h}_N/l_\nu$
- inverse slope $Ct = \cot\beta$
- Kapitza number $\Gamma = \sigma/\left[\rho\nu^{4/3}(g\sin\beta)^{1/3}\right] = (l_c/l_\nu)^2$ with $l_c = \sqrt{\sigma/(\rho g\sin\beta)}$

length scale $\bar{h}_N$ $\rightsquigarrow$parameters :

- Reynolds number $R \equiv \bar{u}_N \bar{h}_N/\nu = q_N = \frac{1}{3} h_N{}^3$
- Weber number $We \equiv \sigma/(\rho g \bar{h}_N^2 \sin\beta) = \Gamma h_N{}^{-2} = (l_c/\bar{h}_N)^2$

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Reduced parameters

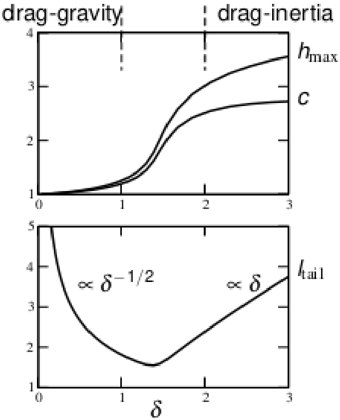
Shkadov notations (Shkadov, 1977)

length scale h_{N} in the y direction
stretched length scale κh_{N} in the x and z directions
κ tuned such that $g \sin\beta$ and $\sigma\partial_{xxx}h$ are of same order
$\rightsquigarrow \kappa = We^{1/3} = (l_c/\bar{h}_{\mathrm{N}})^{2/3}$ with $l_c = \sqrt{\sigma/(\rho g \sin\beta)}$

- reduced Reynolds number $\delta = h_{\mathrm{N}}^3/\kappa = 3R/\kappa$ [3] which measures inertia
- viscous dispersion parameter $\eta = 1/\kappa^2 \ll 1 = (\bar{h}_{\mathrm{N}}/l_c)^{4/3}$ compares viscous and capillary dampings
- reduced inverse slope $\zeta = Ct/\kappa$

[3] In fact this definition is 45 times Shkadov's original definition

Solitary waves: two regimes



Velocity c, amplitude h_{max} and length of the upstream tail, l_{tail}

Question

How to obtain simple equations to handle both regimes ?

- long-wave expansion $\varepsilon = \bar{h}_{\mathrm{N}}/\lambda \ll 1$ assumes inertia is only a corrective term.
- difficulty arises in drag-inertia regime (capillary roll waves)
- need to introduce at least two degrees of freedom h (kinematic) and $q = \int_0^h u\,dy$ (dynamic)
- We limit opurselves here to the simplest two-equation model formulations. . .

Ellipse Velocity Profile formulation

Most of the material of this presentation is published in :

Modelling falling lm ow: an adjustable formulation Sanghasri Mukhopadhyay, Christian Ruyer-Quil, and R. Usha *J. Fluid Mech.* **952** (2022) R3.

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- Nusselt solution

$$u = h^2\left(\frac{y}{h} - \frac{1}{2}\frac{y^2}{h^2}\right) \equiv h^2 g_0(\bar{y}) \qquad \text{where} \quad \bar{y} = y/h\,.$$

- Variation around Nusselt profile

$$\begin{aligned} u &= u^{(0)}(h,q) + \varepsilon u^{(1)}(h,q) \\ &\equiv h^2 g_0(\bar{y}) + \left(\frac{q}{h} - \frac{h^2}{3}\right) f_A(\bar{y}) + \varepsilon u^{(1)}(h,q)\,, \end{aligned}$$

- flow rate definition

$$q = \int_0^h u\, dy\,, \quad \int_0^h u^{(0)}(h)\, dy = \frac{h^3}{3}$$

$$\int_0^1 f_A(\bar{y}) d\bar{y} = 1\,, \qquad f_A(0) = 0\,, \qquad \frac{df_A}{d\bar{y}}(1) = 0\,,$$

- gauge condition

$$\int_0^h u^{(1)}(h,q) dy = 0$$

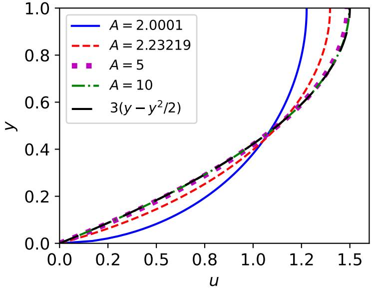
Introduction
EVP formulation
Conclusion

Ellipse profile

$$f_A(\bar{y}) = K_p \left[\sqrt{A^2 - 4(\bar{y}-1)^2} - B \right]$$

with

$$K_p = \frac{1}{\frac{A^2}{4} \sin^{-1}\left(\frac{2}{A}\right) - \frac{1}{2}B}, \qquad B = \sqrt{A^2 - 4}\,.$$



$$\begin{aligned}
\varepsilon\delta\left(u_t + u\,u_x + v\,u_y\right) &= b(h) + u_{yy} + \varepsilon^2\eta\left(2u_{xx} + \left[u_x|_{y=h}\right]_x\right),\\
u_x + v_y &= 0, \qquad u|_{y=0} = v|_{y=0},\\
u_y|_{y=h} &= \varepsilon^2\eta\left(4h_x u_x|_{y=h} - v_x|_{y=h}\right),\\
v|_{y=h} &= h_t + u|_{y=h}h_x,
\end{aligned}$$

where $b(h) = 1 - \varepsilon\zeta h_x + \varepsilon^3 h_{xxx}$

$$\int_0^h g_0 u_{yy}^{(1)} dy = \frac{1}{2}u_y^{(1)}|_{y=h} - \int_0^h u^{(1)} dy = \varepsilon\eta\left(2h_x u_x^{(0)}|_{y=h} - \frac{1}{2}v_x^{(0)}|_{y=h}\right).$$

- EVP formulation

$$
\begin{aligned}
h_t &= -q_x\,, \\
\delta S\, q_t &= \delta \left[\left(G\frac{q^2}{h^2} - G_1\, q\, h - G_2\, h^4 \right) h_x - \left(F\frac{q}{h} + F_1\, h^2 \right) q_x \right] \\
&\quad + \frac{h}{3}\left[b(h) - \frac{3q}{h^2} \right] + \eta \left[\left(J\frac{q}{h^2} - J_1\, h \right) h_x^2 \right. \\
&\quad \left. - K\frac{q_x\, h_x}{h} - \left(L\frac{q}{h} + L_1\, h^2 \right) h_{xx} + M q_{xx} \right],
\end{aligned}
$$

- using $q = \frac{1}{3}h^3 + O(\varepsilon)$ we get (EVPM formulation)

$$
\begin{aligned}
\delta S\, q_t &= \delta \left[(G - 3\,G_1 - 9\,G_2)\,\frac{q^2}{h^2}\, h_x - (F + 3\,F_1)\,\frac{q}{h}\, q_x \right] \\
&\quad + \frac{h}{3}\left[b(h) - \frac{3q}{h^2} \right] + \eta \left[(J - 3\,J_1)\,\frac{q}{h^2}\, h_x^2 - K\frac{q_x\, h_x}{h} \right. \\
&\quad \left. - (L + 3\,L_1)\,\frac{q}{h}\, h_{xx} + M q_{xx} \right].
\end{aligned}
$$

Solitary wave speed

- moving frame $\xi = x - c\,t$
- mass balance gives $q = c\,h + q_0$ where $q_0 = \int_0^h (u - c)\,dy$
- three dimensional dynamical system

$$\frac{dV}{d\xi} = f(V)\,, \quad \text{with} \quad V = (h, h', h'')^t$$

- limit $\delta \gg 1$, introduce slow variable $\tilde{\xi} = \xi / \delta$

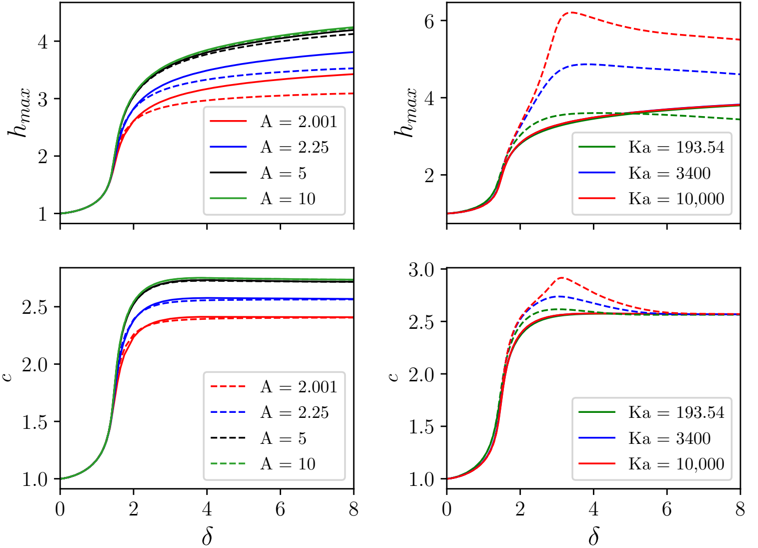
$$\mathcal{N}(h,c)h' \approx \underbrace{\left[h - \frac{3q}{h^2}\right]}_{\mathcal{H}(h,c)}$$

- c_∞ given by solving $\mathcal{N}(h,c) = \mathcal{H}(h,c) = 0$

location $h_{\mathrm{II}} = -1/2 + \sqrt{3(c - 1/4)}$ corresponds to the critical level at which inertia terms cancel out, which gives

$$c^2 + \left(\frac{G - 3\,G_1 - 9\,G_2}{S}\right)\frac{h_{\mathrm{II}}^4}{9} - c\left(\frac{F + 3\,F_1}{S}\right)\frac{h_{\mathrm{II}}^2}{3} = 0$$

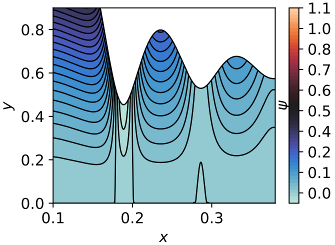
Adjusting c_∞ to its predition 2.560 found by DNS, gives $A = A^\star \approx 2.23219$.



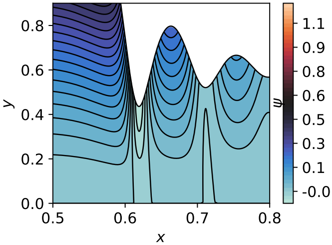
Influence of A

EVPM versus DNS[4]

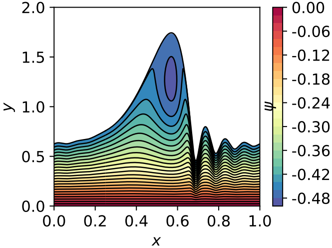
[4]Chakraborty et al. JFM (2014) **745** pp. 564-591



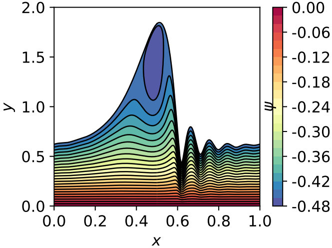
(EVPM) moving frame



DNS moving frame



(EVPM) moving frame



DNS moving frame

Vertical wall water-DMSO mixture $Re = 15$, $f = 16$ Hz, $\mu = 3.13 \times 10^{-3}$Pa.s, $\rho = 1098.3$ kg/m^3, $\sigma = 0.0484$ N/m

Conclusion

- adjustable velocity profiles enable to adjust wave properties
- direct link between velocity profiles and models (compatibility conditions)
- useful alternative to CMA (less algebra)