Stability of non newtonian fluid flows down a slope

GDR TransInter Phase II, Aussois, 10-12 june 2025

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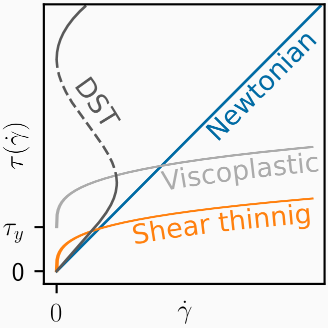
Introduction

Figure 1: Roll waves on a chocolate fountain. Surface coating. Oroville dam spillway (2016). Landslide in Colorado (2014).

Introduction

Same configuration, various rheologies:

- Newtonian (water),
- Shear thinning / Viscoplastic (mud),
- Discontinuous Shear Thickening (cornstarch),
- ...



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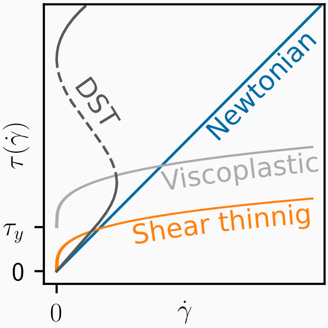
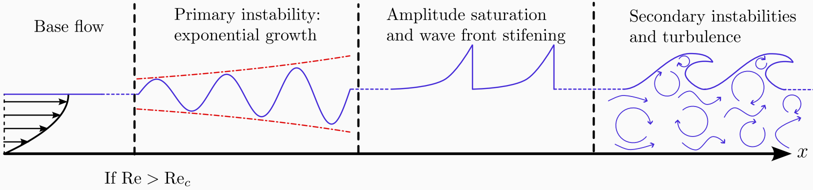


Growth scenario:



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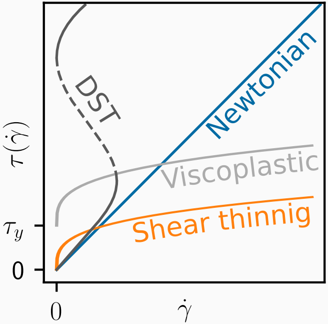

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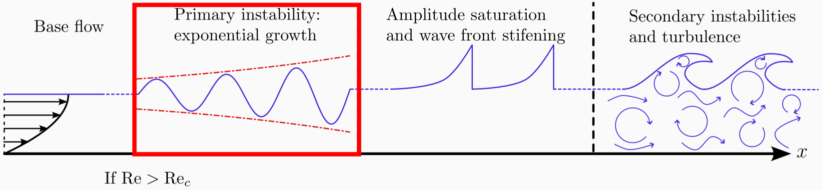

Our focus today: primary instability

Some studies on the primary instability (very non exhaustive)

Newtonian fluids:

- Kapitza (ZETF, 1948)
- Yih (PoF, 1963)
- Benney (JMP, 1966)
- Liu & Gollub (JFM, 1993)
- Ruyer Quil & Manneville (Eur. Phys. J. B, 1998)
- ...

Viscoplastic fluids:

- Balmforth & Liu (JFM, 2004)
- Noma *et al.* (JFM, 2021)
- ...

Shear thinning fluids:

- Ng & Mei (JFM, 1994)
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Discontinuous shear thickening fluids:

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Other (funky) rheologies:

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Is it possible to find a common framework to unify these studies ?

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Is it possible to find a common framework to unify these studies ?

Answer:

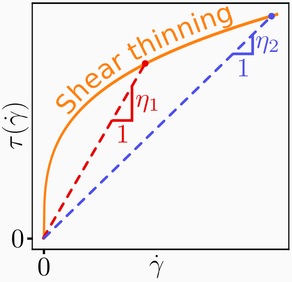
Yes, if we are able to find a common framework for the fluid rheologies.

The common framework: generalised Newtonian fluids

Generalised Newtonian fluids: viscosity

Newtonian fluid:

$\tau(\dot{\gamma}) = \eta\dot{\gamma} \rightarrow \eta =$ constant viscosity



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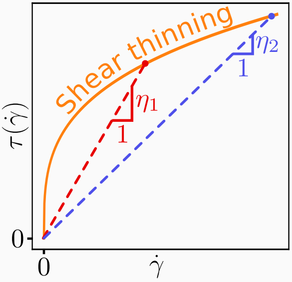

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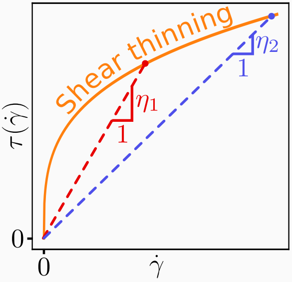

Fluids mentioned earlier are in fact generalised Newtonian fluids:

- Shear thinning fluid: $\eta(\dot{\gamma}) = \kappa\dot{\gamma}^{n-1}$
- Viscoplastic fluid: $\eta(\dot{\gamma}) = \dfrac{\tau_y}{\dot{\gamma}} + \kappa\dot{\gamma}^{n-1}$
- Discontinuous shear thickening: $\eta(\dot{\gamma}) = \dfrac{\eta_s}{(\Phi_J(\tau) - \Phi)^2}$

Generalised Newtonian fluids: fluidity

Express $\dot{\gamma}$ as a function of τ:

$$\dot{\gamma} = \phi(\tau)\tau$$

- $\phi(\tau)$: fluidity function.

Fluidity is the inverse of viscosity:

$$\phi(\tau) = \frac{1}{\eta(\dot{\gamma})}.$$

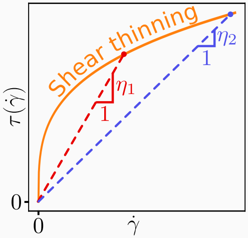

Figure 2: Viscosity

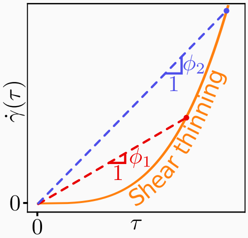

Figure 3: Fluidity

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Rheological laws using fludity:

- Shear thinning fluid: $\phi(\tau) = \frac{1}{\tau}\left(\frac{\tau}{\kappa}\right)^{1/n}$,
- Viscoplastic fluid:
  $\phi(\tau) = \frac{1}{\tau}\left(\frac{\tau - \tau_y}{\kappa}\right)^{1/n-1}$,
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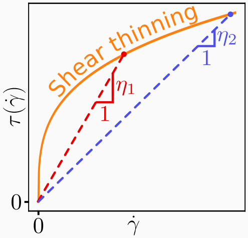

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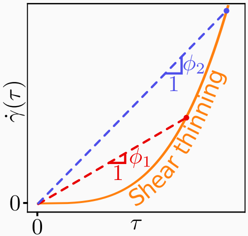

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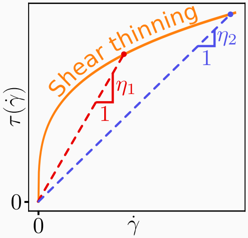

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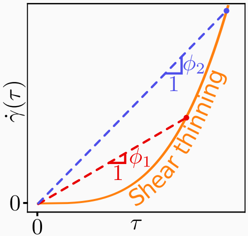


Figure 3: Fluidity

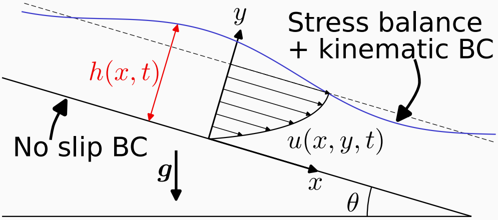
Strategy for a generalised model:
keep η and ϕ known but unspecified.

Orr-Sommerfeld equations for a generalized Newtonian fluid

Incompressible Navier-Stokes equations

2D, incompressible, isothermal flow:

- density: ρ,
- velocity: $\boldsymbol{u} = (u, v)$,
- pressure: p,
- shear stresses: τ_{xx} and τ_{xy},
- film thickness: $h(x,t)$,
- surface tension: σ.



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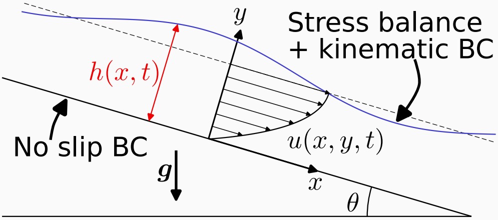


Navier-Stokes equations:

$$\partial_x u + \partial_y v = 0,$$

$$\rho(\partial_t u + u\partial_x u + v\partial_y u) = -\partial_x p + \partial_x \tau_{xx} + \partial_y \tau_{xy} + \rho g \sin\theta,$$

$$\rho(\partial_t v + u\partial_x v + v\partial_y v) = -\partial_y p + \partial_x \tau_{xy} - \partial_y \tau_{xx} - \rho g \cos\theta,$$

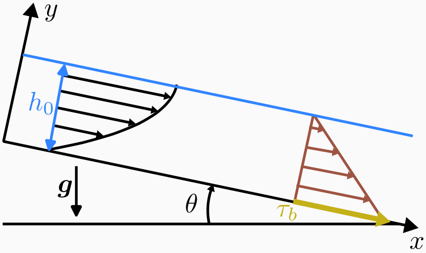
$+$ boundary conditions

with $\phi(\tau)\tau_{xx} = 2\partial_x u$ and $\phi(\tau)\tau_{xy} = \partial_y u + \partial_x v$.

Characteristic scales and dimensionless numbers

"Natural" scales:

- undisturbed film thickness: h_0,
- bottom shear stress:
  $\tau_b = \rho g h_0 \sin\theta$.



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Rheology based scales:

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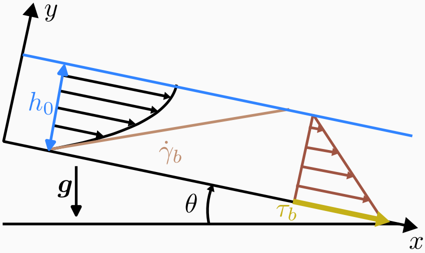

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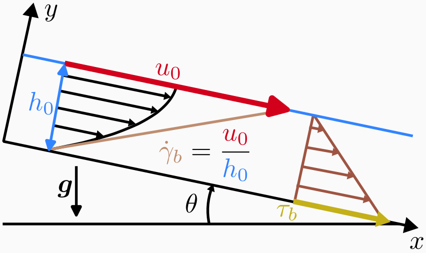

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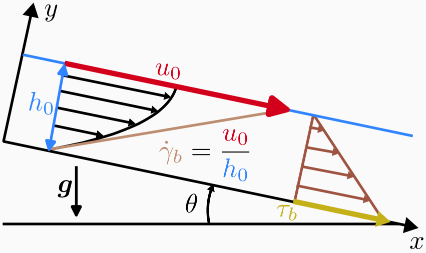

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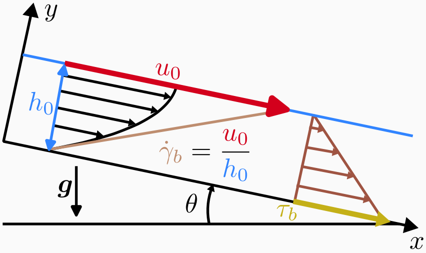

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Dimensionless numbers:

$$\mathrm{Re} = \frac{\rho u_0 h_0}{\eta(\dot\gamma_b)} \quad \text{and} \quad \mathrm{Bo} = \frac{\rho g {h_0}^2 \sin\theta}{\sigma},$$

+ rheology based numbers.

Dimensionless equations

Dimensionless N.S equations:

$$
\begin{aligned}
\partial_x u + \partial_y v &= 0, \\
\mathrm{Re}(\partial_t u + u\partial_x u + v\partial_y u) &= -\mathrm{Re}\partial_x p + \partial_x \tau_{xx} + \partial_y \tau_{xy} + 1, \\
\mathrm{Re}(\partial_t v + u\partial_x v + v\partial_y v) &= -\mathrm{Re}\partial_y p + \partial_x \tau_{xy} - \partial_y \tau_{xx} - \cot\theta, \\
&+ \text{boundary conditions.}
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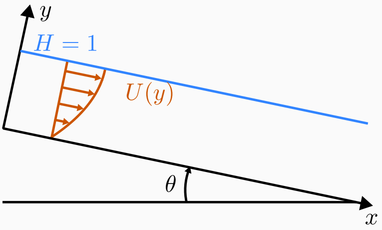
$+$ boundary conditions.

Base flow assumptions:

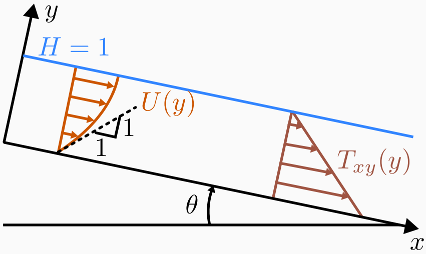
- steady state: $\partial_t = 0$,
- constant thickness: $h(x,t) = H = 1$,
- parallel to the wall: $\boldsymbol{U}(x,y) = (U(y), 0)$,



Base flow

Simplified equations:

$$\partial_y T_{xy} = -1,$$
$$\mathrm{Re}\partial_y P = -\cot\theta,$$
$$\partial_y U = \phi(T_{xy})T_{xy},$$
$+$ boundary conditions.



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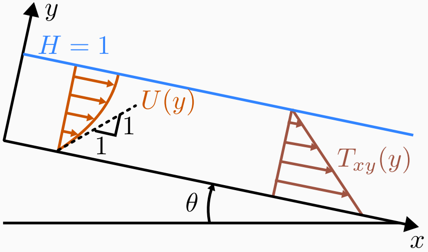

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$$T_{xy}(y) = 1-y, \quad \mathrm{Re}P(y) = (1-y)\cot\theta, \quad U(y) = \int_0^y \phi\,(1-y_1)\,(1-y_1)\,\mathrm{d}y_1.$$

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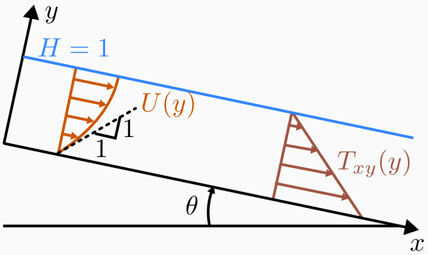

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$U(y)$ depends only on the rheology which is known!

Perturbation equations

Small pertubations around the base flow:

$$u = U + \tilde{u}, \quad v = \tilde{v}, \quad p = P + \tilde{p}, \quad h = 1 + \tilde{h}, \quad \tau = T + \tilde{\tau}, \quad \dot{\gamma} = \dot{\Gamma} + \tilde{\dot{\gamma}}.$$

Perturbations $\rightarrow$ N.S equations $\rightarrow$ linearisation

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Perturbations → N.S equations → linearisation

Linearised N.S equations for the perturbations:

$$\begin{aligned} \partial_x \tilde{u} + \partial_y \tilde{v} &= 0, \\ \mathrm{Re}(\partial_t \tilde{u} + U \partial_x \tilde{u} + \tilde{v} \partial_y U) &= -\mathrm{Re} \partial_x \tilde{p} + \partial_x \tilde{\tau}_{xx} + \partial_y \tilde{\tau}_{xy}, \\ \mathrm{Re}(\partial_t \tilde{v} + U \partial_x \tilde{v}) &= -\mathrm{Re} \partial_y \tilde{p} + \partial_x \tilde{\tau}_{xy} - \partial_y \tilde{\tau}_{xx}, \end{aligned}$$

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Stream function and normal modes decomposition:

$$\tilde{u} = \partial_y \Psi \quad \text{and} \quad \tilde{v} = -\partial_x \Psi,$$

$$\Psi = \psi(y) \exp\left(i\alpha(x - ct)\right),$$

with α: wavenumber, c: phase velocity.

Perturbation equations

Orr-Sommerfeld equation:

$$i\alpha \mathrm{Re}\left((U-c)(\psi''+\alpha^2\psi)-U''\psi\right) =$$
$$-\left(\frac{\psi''}{U''}\right)'' - \alpha^2\left(\left(\frac{\psi}{U''}\right)''+4\left(\frac{\psi'}{\phi}\right)'+\frac{\psi''}{U''}\right)-\alpha^4\frac{\psi}{U''}.$$

+ boundary conditions.

Depends only on the fluidity through $U=\int_0^y \phi(1-y_1)(1-y_1)\mathrm{d}y_1$.

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- long waves ($\alpha \to 0$),
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Numerical resolution:

- dispersion curves,
- comparison with experimental results.

Asymptotic resolution at long waves

Long wave expansion

Temporal stability: $\alpha \in \mathbb{R}$ fixed, ψ and $c \in \mathbb{C}$ variables

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Power series in α:

$$c = c_0 + \alpha c_1 + \alpha^2 c_2 + \dots \quad \text{and} \quad \psi = \psi_0 + \alpha \psi_1 + \alpha^2 \psi_2 + \dots$$

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0th order:

$$\boxed{c_0 = 1 \rightarrow c_0^{dim} = u_0 = \dot{\gamma}_b h_0,}$$

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Power series in α:

$$c = c_0 + \alpha c_1 + \alpha^2 c_2 + \dots \quad \text{and} \quad \psi = \psi_0 + \alpha\psi_1 + \alpha^2\psi_2 + \dots$$

0^{th} order:

$$\boxed{c_0 = 1 \to c_0^{dim} = u_0 = \dot{\gamma}_b h_0,}$$

1^{st} order:

$$\boxed{\mathrm{Re}_c = \frac{1-2q}{1-4q+2\left(\mathcal{M}+\mathcal{K}\right)}\cot\theta,}$$

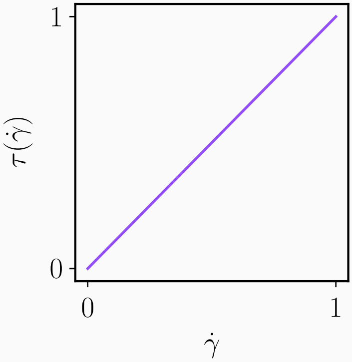
with $q = \underbrace{\int_0^1 U(y)\mathrm{d}y}_{\text{flow rate}}$, $\mathcal{M} = \underbrace{\int_0^1 U^2(y)\mathrm{d}y}_{\propto\text{shape factor}}$ and $\mathcal{K} = \underbrace{\int_0^1\int_0^y\int_0^{y_1}\left(U'(y_2)\right)^2\mathrm{d}y_2\mathrm{d}y_1\mathrm{d}y}_{\propto\text{disipation rate?}}$.

Depoilly *et al.* (PLOS One, 2024)

Comparison with experimental results

Newtonian fluid:

$$\tau(\dot{\gamma}) = \dot{\gamma}$$



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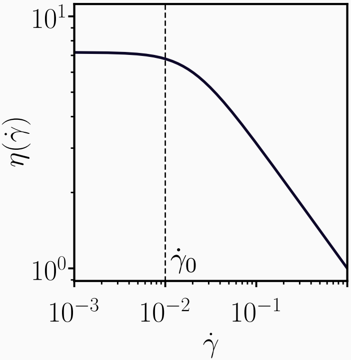
Carreau fluid (shear-thinning):

$$\tau\left(\dot{\gamma}\right) = \left(\frac{1 + (\dot{\gamma}/\dot{\gamma}_0)^2}{1 + (1/\dot{\gamma}_0)^2}\right)^{\frac{n-1}{2}} \dot{\gamma}$$



Carreau (Trans. Soc. Rheol., 1972)

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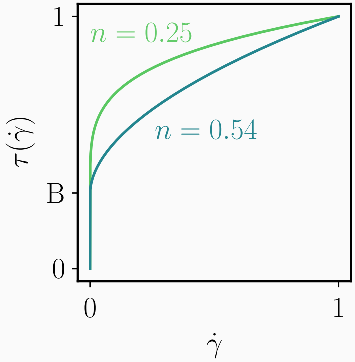

Herschel & Bulkley (Kolloid-Zeitschrift, 1926)

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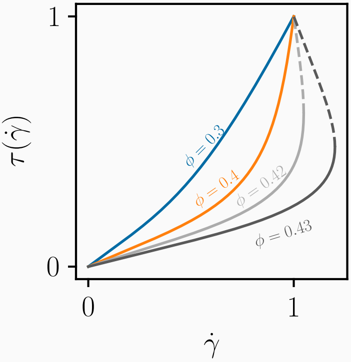
Wyart-Cates law for shear-thickening suspensions:

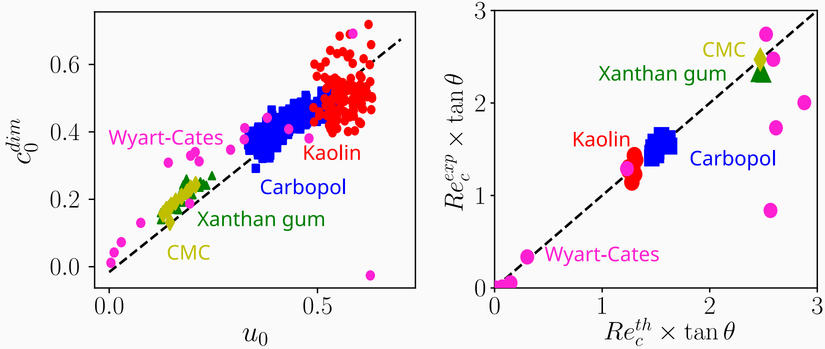
$$\dot{\gamma}(\tau) = \left(\frac{\Phi_J(\tau) - \Phi}{\Phi_J(1) - \Phi}\right)^2 \tau,$$

Φ: volumic fraction, Φ_J : jamming fraction.



Wyart & Cates (PRL, 2014)

Comparison with experimental results



Noma *et al.* (JFM, 2021)
Darbois-Texier *et al.* (JFM, 2023)
Allouche *et al.*(JFM, 2017)

Numerical resolution and dispersion curves

Dispersion curves

Spatial stability: Pulsation $\omega = \alpha c \in \mathbb{R}$ fixed, ψ and $\alpha \in \mathbb{C}$ variables
α_R: wavenumber, $-\alpha_I$: growth rate.

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  $\rightarrow$ for Newtonian and Herschel-Bulkley fluids

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Shooting method:

- easier for spatial stability (polynomial eigenavlue problem),
- well suited for stiff problems.

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$$\alpha\left(U\left(1\right)+\psi\left(1\right)\right)-\omega=0$$

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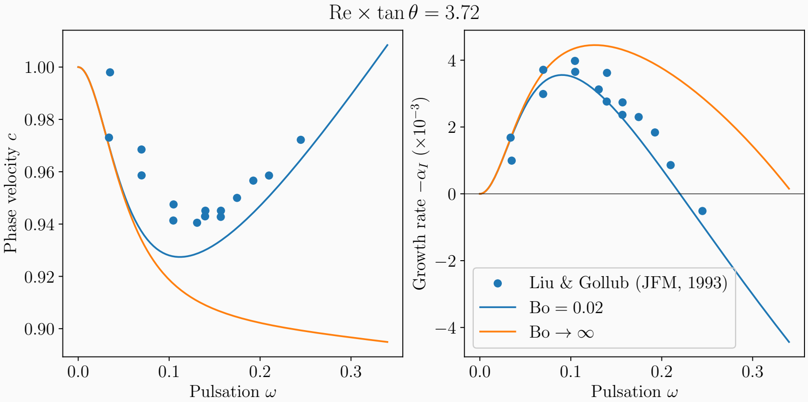
- repeat for multiple ω.

Dispersion curves for a Newtonian fluid



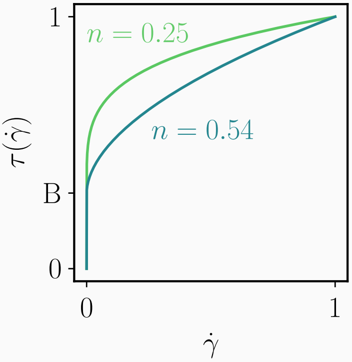
Liu & Gollub (JFM, 1993)

Dispersion curves for Herschel-Bulkley fluids

Herschel-Bulkley rheological law:

$$\tau(\dot{\gamma}) = \mathrm{B} + (1 - \mathrm{B})\dot{\gamma}^n$$

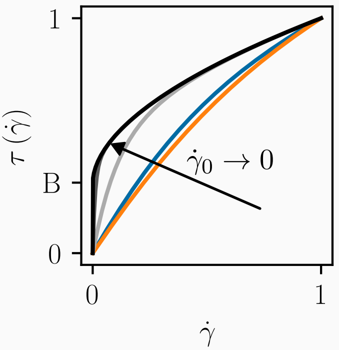
- B: dimensionless yield stress,
- $n \leq 1$: rheological index.



Diverging viscosity at $\dot{\gamma} \to 0$!

Double regularized Herschel-Bulkley rheological law:

$\tau(\dot{\gamma}) = \mathrm{B}$ $+(1-\mathrm{B})$ $\dot{\gamma}^n$

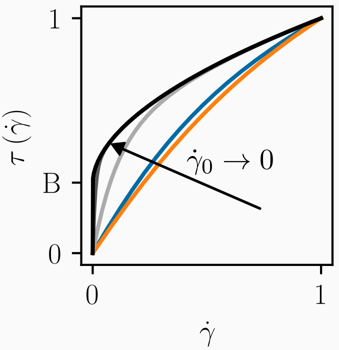


Dispersion curves for Herschel-Bulkley fluids

Double regularized Herschel-Bulkley rheological law:

$\tau(\dot\gamma) = \mathrm{B}\left(1 - \exp\left(-\dot\gamma/\dot\gamma_0\right)\right) + (1 - \mathrm{B})$ $\dot\gamma^n$

- Papanastasiou-like regularization for the yield stress,



Papanastasiou (Journal of Rheology, 1987)

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- Papanastasiou-like regularization for the yield stress,
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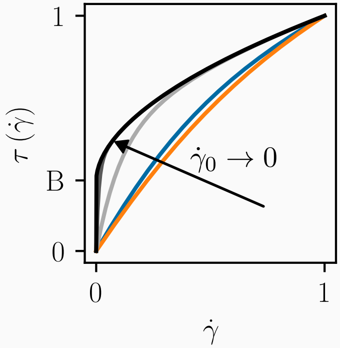

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- Papanastasiou-like regularization for the yield stress,
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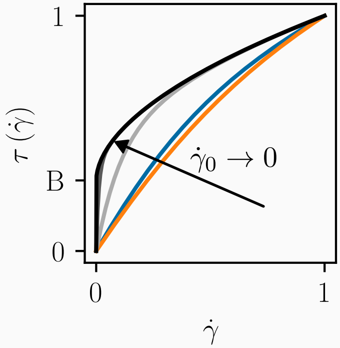
- $\dot\gamma_0$: regularization parameter, no physical meaning.



Choose $\dot\gamma_0$ as small as possible!

Papanastasiou (Journal of Rheology, 1987)
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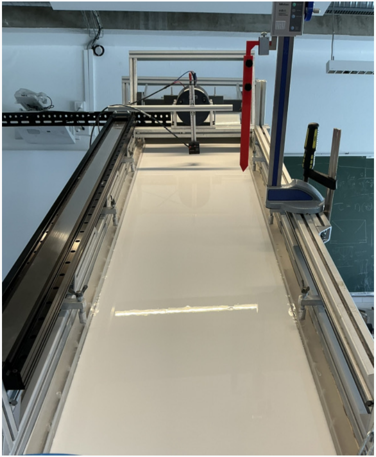
Experimental wave-detection

Experimental setup:

- angle $\theta \approx 1^\circ - 30^\circ$,
- imposed pulsation ω,
- fluid used: Carbopol 980 (Lubrizol),



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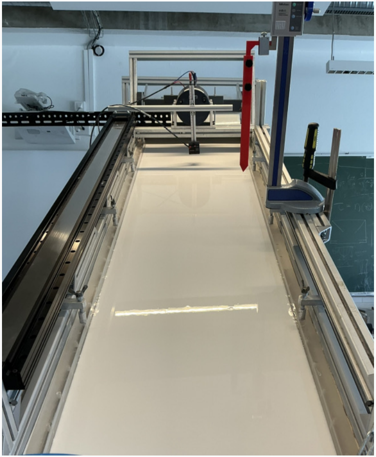
Measurement of α:

- translating laser at velocity v:

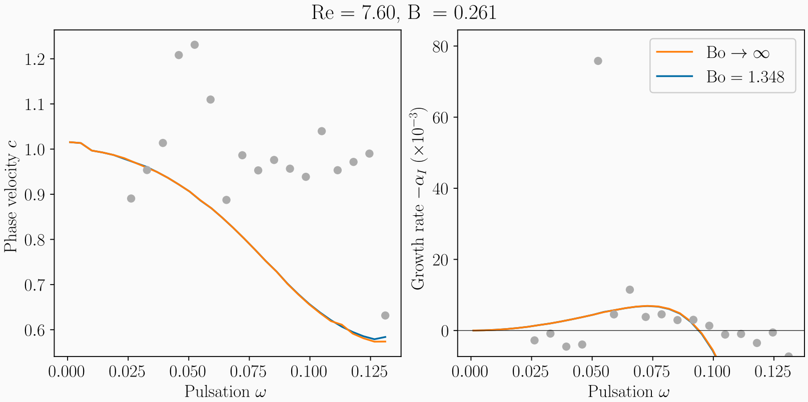
$$s(t) \propto \cos\left((\omega - \alpha_R v)t\right) \exp\left(-\alpha_I v t\right)$$

Doppler effect $\rightarrow$ Fourier analysis $\rightarrow$ α.



Yoshida *et al.* (Jour. of Phys. E, 1981)

Dispersion curves for viscoplastic fluids



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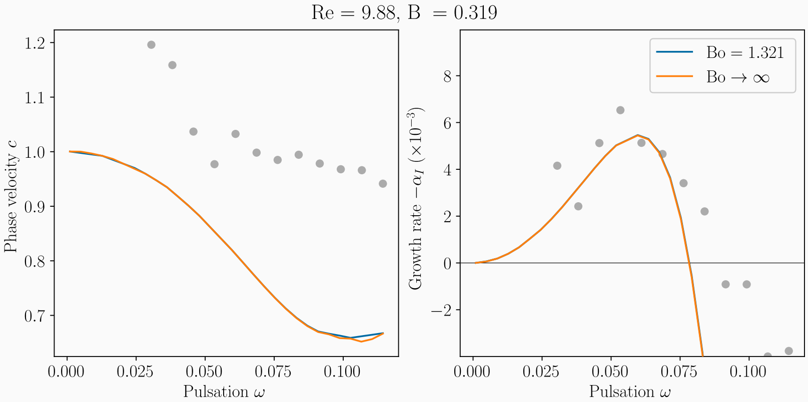

Conclusion

Generalized model:

- generalised Newtonian fluids: $\tau(\dot{\gamma}) = \eta(\dot{\gamma})\dot{\gamma}$ or $\dot{\gamma} = \phi(\tau)\tau$,
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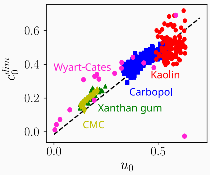
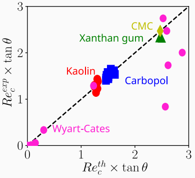
- wave celerity c_0 and stability threshold Re_c,
- good agreement with experimental results for all rheologies found in literature.





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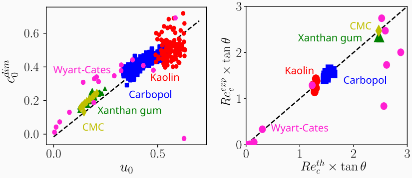
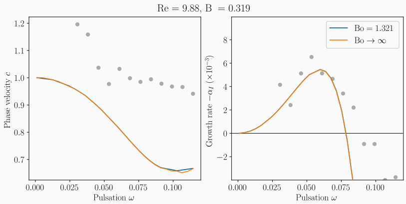


Dispersion curves:

- $\alpha = f(\omega, \mathrm{Re})$,
- good agreement for the growth rate,
- trickier to predict wave velocity.



Perspectives

More experiments:

- explore new rheologies (Carreau, . . .),
- further validation of our model.

Generalise other models:

- Benney equation, integrated models,
- non-linear stability.

Add more funky effects:

- viscoelasticity.

Thank you very much !